ELSEVIER Journal of Geometry and Physics 16 (1995) 345-384

JOURNAL OF GEOMETRY AND PHYSICS

Symplectic geometry of the loop space of a Riemannian manifold

Tilmann Wurzbacher [1]

Ruhr-Universität Bochum, Fakultät für Mathematik, Universitätsstrasse 150, 44780 Bochum, Germany

Received 22 February 1994; revised 19 July 1994

Abstract

The geometry of the "moment" map of the action of the diffeomorphism group of the circle on loop spaces is studied in detail. We also develop aspects of the general symplectic geometry of loop spaces and deduce e.g. a Hamiltonian description of nonlinear Sigma-models.

Keywords: Infinite dimensional symplectic geometry; Diffeomorphism action on loops;
1991 MSC: 58 B 20, 58 D 19

0. Introduction

The last two decades have seen a renaissance of the interplay between mathematics and physics and quite a prominent role in it is played by loop spaces. They are the physical configuration spaces of "closed bosonic strings", i.e. loops in a (pseudo-)Riemannian manifold. Attempts to derive properties of "quantized string field theories" geometrically led physicists to the idea of applying geometric quantization techniques to the loop space (see [Bo-Ra] and references therein).

Other quantum field theories like "super-symmetric nonlinear Sigma-models" also involve the study of loop spaces, especially their Dirac-operators (see [Benn] and [Seg] and references therein). Again the symplectic geometry of loop spaces seems to be important for the geometric understanding (see [At]).

Motivated by the goal to justify and generalize the geometric quantization approach to the quantization of strings one should always work Diff(S^1)-equivariantly (Diff(S^1)

[1] Partially supported by SFB 237 in Bochum. Current address: IRMA, Université L. Pasteur, 7, rue René Descartes, 67084 Strasbourg cedex, France.

SSDI 0393-0440(94)00033-6

acts by reparametrizing maps from the circle to the manifold): physically spoken strings are demanded "to have no internal structure", mathematically one is, after all, interested in varieties, here closed arcs, rather than in the parametrizing maps.

The present work develops rigorously the classical symplectic geometry of the $\mathrm{Diff}(S^1)$-action on loop spaces—considering it as an important example of infinite-dimensional geometry. The results can be summarized by saying that the $\mathrm{Diff}(S^1)$-action comes with a smooth "moment" map Φ, which has most of the properties classical moment maps enjoy in the finite-dimensional setting—despite the fact that the (weakly) symplectic form has degeneracy and the action is neither differentiable in any strong sense nor proper.

More concretely, it is e.g. shown that the kernel of the differential $D\Phi$ is the skew-complement to the orbit and the image of $D\Phi$ in a point γ is the annihilator of the isotropy algebra of γ. Furthermore we find a surprisingly simple geometric description of the generic orbit structure: the Φ-fibres and the orbits intersect in a circle and their tangent spaces add up, together with a one-dimensional slice, to the whole tangent space. This enables us to describe the reduced spaces, i.e. the "symplectic quotient of the loop space by the action".

The article is organized as follows. The first section describes the symplectic structure and the $\mathrm{Diff}(S^1)$-action, considering carefully the crucial differentiability problems, which are typical for infinite-dimensional situations. In the second section we derive the coadjoint action of $\mathrm{Diff}(S^1)$, define the moment map and prove its smoothness. In Section 3 we prove the above sketched results about Φ, comparing them with the classical situation recalled in Subsection 2.4. We stress that these results can *not* be proven by simply extending all the notions to infinite dimensions.

The fourth and last section consists of several remarks which involve the symplectic structure of the loop space. For example, in Subsection 4.4 we give a symplectic description of non-linear Sigma-models: a Hamiltonian formulation for a classical field theory built from the set of maps from a cylinder to a Riemannian manifold M and the harmonic map equation as the governing field equation ("Sigma-models"). The Hamilton function in this picture consists of the fibre norm (of the tangent bundle of the loop space of M), which should give rise to a Laplacian after quantization—and the Hamiltonian generator of the reparametrization by rotation of a loop. This is the classical counterpart of the use of equivariant Laplacians in the (partly heuristical) attempts to quantize Sigma-models (see [Benn] and references therein).

1. The weakly symplectic structure

1.1. Spaces of loops

We will consider an arbitrary Riemannian manifold (M, g), which we neither assume to be compact nor to be complete, and will denote its Levi-Civita connection on TM and associated bundles by ∇.

The space of smooth loops, $C^\infty(S^1, M)$ with its standard structure of a Frechet manifold (see [H]) will be called X_∞. Several completions to Banach manifolds of mappings are possible, most notably $X_k := H^{k,2}(S^1, M)$, the space of (Sobolev) maps whose kth (covariant) derivative is square integrable with respect to the unique flat metric on the circle S^1 inducing the bi-invariant measure of total mass 1 on S^1. (We identify S^1 with $[0,1]/\sim$ and denote elements of $I := [0,1]$ by t, the above mentioned measure by dt and the dual vector field by d/dt).

For $k \in \mathbb{N}^{\geq 1}$ these spaces are smooth Hilbert manifolds modelled on $H^{k,2}(S^1, \mathbb{R}^d)$, where $d = \dim_{\mathbb{R}} M$(see e.g. [E].) In the sequel we will sometimes use the letter X for X_∞ and X_k, if a distinction is not necessary. We would like to remark that for reasons coming from physics and measure theory it seems to be desirable to work with loops of the "Sobolev quality $H^{1/2,2}$", but since a function in $H^{1/2,2}(S^1, \mathbb{R})$ is not even necessarily essentially bounded it is not possible to define a smooth manifold structure on $H^{1/2,2}(S^1, M)$, which can nevertheless be given set-theoretically by embedding M into $\mathbb{R}^n$.

What is possible and useful is to define certain completions of the tangent bundle of X. Let us recall the following identifications: For $\gamma \in X_\infty, T_\gamma X_\infty = \Gamma_{C^\infty}(S^1, \gamma^*TM)$, i.e., a tangent vector to γ is a vector field along the map γ.

Denoting the induced connection on γ^*TM by ∇/dt, we get the following norms for a measurable section of γ^*TM:

$$\|u\|_k^2 := \int_0^1 g_{\gamma(t)}(u(t), u(t))dt + \sum_{j=1}^{k} \int_0^1 g_{\gamma(t)} \left(\left(\frac{\nabla}{dt}\right)^{(j)} u(t), \left(\frac{\nabla}{dt}\right)^{(j)} u(t) \right) dt .$$

It follows that $T_\gamma X_k = \Gamma_{H^{k,2}}(S^1, \gamma^*TM)$.

Obviously TX_k can now be completed fibrewise with respect to any weaker norm than $\| \ \|_k$, most notably we introduce for all k and ∞ the "L^2-metric" on $T_\gamma X$:

$$\langle u, v \rangle_0 := \int_0^1 g_{\gamma(t)}(u(t), v(t)) .$$

The completion of TX with the L^2-metric will be called $L(X)$ and sections of it are referred to as "L^2-vector fields" on X.

We would like to introduce, painfully enough, another bundle, namely the "good" cotangent bundle of X_∞:

$$\tilde{T}_\gamma^* X_\infty := \Gamma_{C^\infty}(S^1, \gamma^*T^*M) ,$$

which by its natural pairing

$$\tilde{T}^*_\gamma X_\infty \times T_\gamma X_\infty \longrightarrow \mathbb{R}, \quad (b,u) \longmapsto \int_0^1 b(t)(u(t))dt$$

should sit in any reasonable cotangent bundle of X_∞. Note that there is no problem in the definition of cotangent bundles in the category of Hilbert manifolds.

1.2. Reparametrization of loops

Being interested in geometry, i.e. in subvarieties more than in the parametrization maps, one would like to work modulo the action of $\mathcal{G}$, the group of orientation preserving diffeomorphisms of the circle. The same goal can be drawn from the physical demand that "strings have no internal structure". The action is given by

$$\vartheta : \mathcal{G} \times X_\infty \longrightarrow X_\infty, \ (g,\gamma) \longmapsto \gamma \circ g^{-1}$$

and is smooth if $\mathcal{G}$ is given its natural Frechet group structure and differentiability is defined in the usual sense for Frechet spaces (see [H] for the relevant definitions). The differential of the map

$$\vartheta_g : X_\infty \longrightarrow X_\infty, \ \gamma \longmapsto \gamma \circ g^{-1}$$

is given by $(\vartheta_g)_{*g}(u) = u \circ g^{-1}$ for $u \in T_\gamma X_\infty$. Going to the Sobolev manifolds X_k the maps ϑ_g pertain to be smooth with the above differential, but the equation

$$(D\vartheta)_{(g,\gamma)}(\xi,u) = u \circ g^{-1} - ((D\gamma) \circ g^{-1}) \cdot ((Dg)^{-1} \circ \xi \circ g^{-1}) \qquad (*)$$

for $\xi \in T_g\mathcal{G} = \Gamma_{C^\infty}(S^1, g^*TS^1)$ and $u \in T_\gamma X$, shows that ϑ is not even C^1 as a map $\mathcal{G} \times X_k \longrightarrow X_k$.

An easy argument with metrics on S^1 and M yields the fact that ϑ is always C^0. We remark that it is possible to work with the groups $\mathcal{G}_k := \{g \in H^{k,2}(S^1,S^1) \mid g$ is bijective and $g^{-1} \in H^{k,2}(S^1,S^1)\}$ for $k \geq 2$ (for $k = 1$, the set $\mathcal{G}_1$ is *not* a group), whose multiplication and inversion are C^0, but not C^1 with respect to their natural Hilbert manifold structure as open subsets of $H^{k,2}(S^1,S^1)$, and in fact we will use appropriate "Sobolev-diffeomorphisms" to prove certain facts, but since these actions

$$\mathcal{G}_k \times X_k \longrightarrow X_k$$

are only C^0 as well we see no point in complicating things. We denote the rotation subgroup of $\mathcal{G}$ by Rot S^1 and remark that—again by Eq. $(*)$—not even Rot S^1 acts C^1 on X_k. Since the Lie algebra $\mathbf{g}$ of $\mathcal{G}$ is isomorphic to $\Gamma_{C^\infty}(S^1,TS^1)$ with (minus) the standard bracket on vector fields (see subsection 2.1 for a summary of this and related facts), we can give an explicit description of the fundamental vector fields of the $\mathcal{G}$-action on X_∞:

$$\mathbf{g} \longrightarrow \Xi(X_\infty) = \Gamma_{C^\infty}(X_\infty, TX_\infty)\,,$$
$$\xi = f(t)\, d/dt \longmapsto \xi_X\,,$$
$$\left[\xi_X(\gamma)\right](t) = -f(t)\dot\gamma(t)\,,$$

where $\dot\gamma(t) = (\gamma_*)_t(d/dt|_t)$. For the generator of the rotations we get especially

$$\left[\zeta_X(\gamma)\right](t) = -\dot\gamma(t)\,.$$

Obviously, these objects do not give rise to section in the tangent bundles of X_k, but in their $H^{k-1,2}$-completions. So we get "generalized L^2-vector fields", which are indeed smooth sections (of the appropriate bundles). See e.g. [Kl] for the smoothness of ζ_X in the case of $k = 1$, the general result follows similarly.

1.3. The canonical 1-form on the loop space of a Riemannian manifold

For a smooth loop γ one can define the following functional on $T_\gamma X_\infty$ (see [At]):

Definition 1.

$$\mu_\gamma(u) := \frac{1}{2}\int_0^1 g_{\gamma(t)}(u(t), \dot\gamma(t))\,dt\,.$$

Remarks 2.

(1) Geometrically $\mu_\gamma(u)$ measures the average of the length of the tangential part of the vector field u along γ weighted by the norm of velocity of γ. This implies obviously that the kernel of μ_γ consists of those vector fields along γ which are orthogonal to $\dot\gamma$ in the L^2-metric:

$$\text{kern}\ \mu_\gamma = \{u \in T_\gamma X_\infty \mid \langle u, \dot\gamma\rangle_0 = 0\}\,.$$

(2) $\mu_\gamma(u)$ is the result of pairing the element $\frac{1}{2}g(\dot\gamma, *)$ of $\tilde T^*_\gamma X_\infty$ with u. These functionals fit together to form a smooth section of $\tilde T^* X_\infty$.

(3) The construction of μ is closely related to the following method of producing differential forms on mapping spaces:

Let Map (N, M) be the space of smooth ($H^{k,2}, C^k$, etc.) maps from a compact oriented n-dimensional manifold N to a manifold M. The evaluation map

$$N \times \text{Map}\ (N, M) \xrightarrow{\text{Ev}} M, \quad (n, f) \longmapsto f(n)$$

yields by pullback differential forms on its *LHS*, fibre integration over N gives us a linear map:

$$\mathcal{E}^q(M) \longrightarrow \mathcal{E}^{q-n}(\text{Map}\ (N, M))\,,$$
$$\Omega^q \longmapsto \int_N \text{Ev}^*(\Omega) =: \omega^{q-n}\,.$$

The group $\mathrm{Diff}(N) \times \mathrm{Diff}(M)$ acts on Map (N, M) and the form ω^{q-n} is invariant under $\mathrm{Diff}^+(N) \times \mathrm{Diff}_\Omega(M)$, where $\mathrm{Diff}^+(N)$ denotes the orientation preserving diffeomorphisms of N and $\mathrm{Diff}_\Omega(M)$ the Ω-preserving diffeomorphisms of M. Furthermore $d\,\omega = \int_N \mathrm{Ev}^*(d\Omega)$.

In the case of loop spaces of three-manifolds this idea was exploited in [Ma-We] and [Br] by taking Ω equal to a volume form on M, giving a closed 2-form on X.

It is also related to Chen's "Iterated Integrals" [Ch], which lead to the calculation of loop space cohomology. In the case at hand we pull back a symmetric 2-tensor, the metric, integrate over a 1-dimensional (!) manifold and are left with a 1-tensor, which is as well a 1-form.

Without relying on the above principles we will make explicit the main invariance properties of μ on X. For this aim we define the following useful map:

$$C^\infty(S^1, S^1) \longrightarrow C^\infty(S^1, \mathbb{R}),\; g \longmapsto \dot{g},$$

where $\dot{g}$ is defined by

$$(g_*)_t \left(d/dt|_t\right) = \dot{g}(t)\, d/dt|_{g(t)}\,.$$

For $g \in \mathcal{G}$ this function $\dot{g}$ is strictly positive on S^1. (We remark that this works as well in the Sobolev cases.) Furthermore we define a section σ of $\wedge^p \tilde{T}^* X_\infty$ to be "invariant" by a diffeomorphism ϕ of X_∞ iff

$$(\phi^*\sigma)_\gamma(u_1, \ldots, u_p) := \sigma_{\phi(\gamma)}(\phi_* u_1, \ldots \phi_* u_p) = \sigma_\gamma(u_1, \ldots, u_p)$$

for all $u_j \in T_\gamma X_\infty$.

Lemma 3. *μ is $\mathcal{G}$-invariant, i.e. $\vartheta_g^* \mu = \mu$ for all g in $\mathcal{G}$.*

Proof. Obvious. □

Definition 4. Given M and g, the isometry group of (M, g) will be denoted by H.

We observe that the actions of $\mathcal{G}$ and H commute!

Lemma 5. *μ is H-invariant.*

Proof. H acts on X_∞ by

$$\theta_h : X_\infty \longrightarrow X_\infty,\; \gamma \longmapsto h \circ \gamma$$

with differential

$$(\theta_h)_*(u)(t) = (Dh)_{\gamma(t)}(u(t))$$

for $u \in T_\gamma X_\infty$. Thus we get

$$\left(\theta_h^* \mu\right)_\gamma (u) = \mu_{h\circ\gamma}((\theta_h)_* u) = \mu_\gamma(u)\,.$$ □

We remark that the action of $\mathcal{L}H$, the group of smooth loops in H, on X_∞ preserves the L^2-product on TX_∞, but not μ!

Going to the category of Sobolev loops, we see that

$$\mu_\gamma(u) = -\tfrac{1}{2}\langle u, \zeta_X(\gamma)\rangle_0$$

and thus μ is a section not only of T^*X_k, but of

$$(L(X))^* \subsetneqq T^*X_k = (TX_k)^* .$$

We close this section with the smoothness of this section:

Lemma 6. *μ is a smooth section of $L(X_k)^*$ for all $k \geq 1$.*

Proof. Let U be a smooth local section of $L(X_k)$. Since ζ_X is a smooth global section of $L(X_k)$ and $\langle , \rangle_0$ a smooth fibre metric on $L(X_k)$ it follows that $\gamma \longmapsto \langle U(\gamma), \zeta_X(\gamma)\rangle_0$ is a smooth function. Thus μ is smooth as well. □

1.4. The weakly symplectic structure of the loop space

Given the canonical 1-form μ on a loop space, we proceed as follows:

Definition 7. The weakly symplectic form on a loop space is defined as

$$\omega := d\mu .$$

Remarks 8.

(1) As in the case of finite dimensions

$$(d\mu)(U,V) := U(\mu(V)) - V(\mu(U)) - \mu([U,V]) ,$$

for U, V smooth, locally defined vector fields on X.

(2) To cast this in the case of X_∞ into the language of sections of vector bundles on X necessitates a suitable notion of tensor products of locally convex complete topological vector spaces, which we do not need in the sequel and therefore omit. In the case of Sobolev loops all bundles are Hilbert bundles and there is no problem of taking tensor bundles. Thus

$$\omega \in \Gamma_{C^\infty}\left(X_k, \wedge^2(T^*X_k)\right) ,$$

and even better (see below).

Lemma 9. *Let γ be in X_∞ and $u, v \in T_\gamma X_\infty$, then*

$$\omega_\gamma(u,v) = \int_0^1 g_{\gamma(t)}\left(\frac{\nabla u}{dt}(t), v(t)\right) dt .$$

Proof. Let u and v be elements in $T_\gamma X_\infty$, i.e. smooth sections of TM along γ. We find an open neighbourhood B of 0 in the 2-dimensional subspace spanned by u and v in $T_\gamma X_\infty$, such that

$$\vartheta : B \longrightarrow X,\ \vartheta(r,s) := \mathrm{Exp}_\gamma(ru+sv)\ ,$$

(given by $(\mathrm{Exp}_\gamma(w))(t) := \mathrm{Exp}_{\gamma(t)}(w(t))$ for $w \in T_\gamma X_\infty$) is a well-defined map with $\vartheta(0,0) = \gamma$ and

$$(\vartheta_*)_{(0,0)}(u) = u, \qquad (\vartheta_*)_{(0,0)}(v) = v\ ,$$

if we consider u and v on the LHS as constant vector fields on $T_\gamma X_\infty$. In the sequel we will consider the following "extensions" of u and v:

$$U := \vartheta_*(u), \qquad V = \vartheta_*(v)\ ,$$

which are in fact vector fields along ϑ. It is furthermore useful to introduce the map

$$\theta : B \times S^1 \longrightarrow M,\ \theta(r,s,t) := \vartheta(r,s)(t)\ .$$

It allows us to describe the velocity field on a curve $\vartheta(r,s)$ by $\theta_*(\partial/\partial t) =: \zeta$ and to identify

$$U = \theta_*\left(\partial/\partial r\right), \qquad V = \theta_*\left(\partial/\partial s\right)\ .$$

By remark 8(1) above, it suffices to find local extensions of u and v in the neighbourhood of γ.

In fact we need to know the extension $\tilde{u}$ of u only in direction of the flow of the extensions $\tilde{v}$ of v and vice versa. Thus $\tilde{u} = U$, $\tilde{v} = V$ do the job. Denoting $\theta(r,0,t)$ by $\gamma_r(t)$ we find:

$$\begin{aligned}
u(\mu(V)) &= \frac{1}{2}\left.\frac{\partial}{\partial r}\right|_{r=0} \int_0^1 \left\{g_{\gamma_r(t)}\left(V_{\gamma_r(t)}, \dot{\gamma}_r(t)\right)\right\} dt \\
&= \frac{1}{2}\int_0^1 \left.\frac{\partial}{\partial r}\right|_{r=0} \left\{g_{\gamma_r(t)}\left(V_{\gamma_r(t)}, \zeta(\gamma_r(t))\right)\right\} dt \\
&= \frac{1}{2}\int_0^1 \{g_{\gamma_r(t)}\left(\nabla_U V(\gamma_r(t))\, ,\, \zeta(\gamma_r(t))\right) \\
&\quad + g_{\gamma_r(t)}\left(V(\gamma_r(t)), \nabla_U \zeta(\gamma_r(t))\right)\}|_{r=0}\, dt \\
&= \frac{1}{2}\int_0^1 \{g_{\gamma(t)}\left(\nabla_U V, \zeta\right) + g_{\gamma(t)}\left(V, \nabla_U \zeta\right)\}\, dt\,.
\end{aligned}$$

Recalling the above definition of $d\mu$ and the following three facts:

- U, V and ζ commute,
- the torsion of ∇ vanishes,
- $(\nabla_\zeta U)(\gamma(t)) = (\nabla u/dt)(t)$

(see e.g. [Ga-Hu-La]) imply straightforwardly

$$(d\mu)_\gamma(u,v) = \int_0^1 g_{\gamma(t)}\left(\frac{\nabla u}{dt}(t),\, v(t)\right) dt. \qquad \square$$

Remarks 10.

(1) The same formula holds for $\gamma \in X_k$, $k \geq 1$.

(2) The lemma shows that it suffices to assume that u is of the quality $H^{1,2}$ and v of quality L^2 (or both of class $H^{1/2,2}$!). Thus we can define a "tensor field" $\breve{\omega}$ by prescribing the order:

$$\breve{\omega}_\gamma(u,v) := \int_0^1 g_{\gamma(t)}\left(\frac{\nabla u}{dt}(t), v(t)\right) dt$$

for all $u \in \overline{T_\gamma X}$, the $H^{1,2}$-completion of the tangent space, and $v \in L(X)_\gamma$, the L^2-completion of the tangent space. This yields a smooth section $\breve{\omega}$ of $(\overline{TX})^* \otimes L(X)^*$, since the map $u \longmapsto \nabla u/dt$ is a smooth linear bundle map $TX \longrightarrow L(X)$ (see e.g. [Kl]). Thus we get for each smooth section ξ of $L(X)$ a smooth 1-form on X by contraction:

$$(i_\xi \breve{\omega})(U) := -\breve{\omega}(U, \xi_X),$$

where U is a locally defined vector field on X.

(3) Since μ is $(\mathcal{G} \times H)$-invariant, the same is true for ω.

Now we will compute the degeneration of the obviously closed 2-form ω, showing that its dimension is always bounded by the dimension of M. Since the loop space is infinite dimensional, this allows to call ω "weakly symplectic".

Lemma 11. *Let γ be in X_∞, then*

$$\mathrm{kern}\, \omega_\gamma = \{u \in T_\gamma X_\infty \mid (\nabla u/dt)(t) = 0 \quad \forall t\}$$

and this space is included in

$$\mathrm{kern}(P_1 - \mathrm{Id}) \subset T_{\gamma(0)} M,$$

where P_1 denotes the parallel transport along γ.

Proof. Let $u \in \mathrm{kern}\, \omega_\gamma$, then

$$0 = \omega_\gamma(u,v) = \int_0^1 g_{\gamma(t)}\left(\frac{\nabla u}{dt}(t), v(t)\right) \quad \forall v \in T_\gamma X_\infty.$$

Setting $v = \nabla u/dt$ implies $\nabla u/dt = 0$. Obviously $\nabla u/dt = 0$ implies that u is in kern ω_γ. Given $\mathrm{u}(0) \in T_{\gamma(0)}M$, parallel transport along γ yields a unique smooth section $u : I = [0,1] \longrightarrow TM$ over γ such that

$$\frac{\nabla u}{dt}(t) = 0 \quad \text{and} \quad u(0) = \mathrm{u}(0)\,.$$

This u is in $T_\gamma X_\infty$ iff $u(0) = u(1)$ to all orders, thus implying especially

$$P_1(u(0)) = u(1)\,. \qquad \square$$

Remarks 12.
(1) We denote kern ω_γ by N_γ in the sequel of this work.
(2) One obviously finds $\dim \operatorname{kern} \omega_\gamma \leq \dim M$ for all $\gamma \in X_\infty$.
(3) For a generic loop $\operatorname{kern} \omega_\gamma = \{0\}$, since $\operatorname{Eig}(P_1, 1) = \{0\}$.
(4) For constant loops $\gamma(t) \equiv \gamma(0)$ and parallel transport is trivial, giving $\operatorname{Kern} \omega_\gamma \simeq T_{\gamma(0)}M$.
(5) The lemma holds verbatim for X_k, with the appropriate changes in the proof.

Though ω has degeneracy, one may still attempt to define a ((Rot S^1)-invariant) "compatible almost complex structure" J on X, i.e. a smooth vector bundle endomorphism J of TX such that

$$J_\gamma^2 = -\operatorname{Id} \bmod (\operatorname{kern} \omega_\gamma)\,, \qquad J_\gamma = 0 \quad \text{on } \operatorname{kern} \omega_\gamma\,,$$
$$\omega_\gamma(u, J_\gamma u) > 0 \quad \text{for all } \gamma \text{ and } u \in T_\gamma X \setminus (\operatorname{kern} \omega_\gamma)\,.$$

This is indeed possible in the flat case by means of Fourier-modes (see [Bo-Ra]) and its pointwise extension to the general case is sketched in [Seg]. We will provide this construction in some detail, since it will be useful in Section 3, and we will comment briefly on its shortcomings. Let $\tilde{H}$ be the pre-Hilbert space $(T_\gamma X)^{\mathbb{C}} = \Gamma(S^1, \gamma^*TM \otimes \mathbb{C})$ with the L^2-scalar product $\langle\ ,\ \rangle_0$. By means of the spectral decomposition of the symmetric operator $A_\gamma := i\nabla/dt$ we define $J_\gamma^{\mathbb{C}} := if(A_\gamma)$, where f is compounded of characteristic functions: $f = -\chi_{\mathbb{R}^{<0}} + \chi_{\mathbb{R}^{>0}}$ (see e.g. [R] for the symbolic calculus of normal operators).

The operator $J_\gamma^{\mathbb{C}}$ extends to a selfadjoint bounded operator on H, the completion of $(\tilde{H}, \langle\ ,\ \rangle_0)$ and respects the real points of H. We denote the restriction of $J_\gamma^{\mathbb{C}}$ to $T_\gamma X_\infty$ (let us concentrate here on the smooth case) by J_γ and summarize its properties:

Lemma 13. *The endomorphism J_γ of $T_\gamma X_\infty$ is continuous with respect to the above pre-Hilbert topology and fulfills:*
(i) $J_\gamma^2 \equiv -\operatorname{Id} \bmod (\operatorname{kern} \omega_\gamma)$,
(ii) $J_\gamma = 0$ *on* $\operatorname{kern} \omega_\gamma$,
(iii) J_γ *is anti-selfadjoint with respect to* $\langle\ ,\ \rangle_0$,
(iv) $\omega_\gamma(J_\gamma u, J_\gamma v) = \omega(u, v)\ \forall u, v \in T_\gamma X_\infty$,
(v) $\omega_\gamma(u, J_\gamma u) > 0\ \forall u \in T_\gamma X_\infty \setminus \operatorname{kern} \omega_\gamma$.

Proof. The claims follow easily by the properties of the spectral measure of the symmetric operator A (see again [R]). □

We remark that the discontinuity of the function f in 0 causes the section $J : X \longrightarrow \mathrm{End}(TX)$ to be not even continuous with respect to the $\langle\ ,\ \rangle_0$-topology on End(TX), not speaking of any derivative. Thus we do not get an almost complex structure in the desired sense (but the weaker notion of a "polarization" seems to be achievable, see [Seg]!).

2. The moment map of the Diff(S^1)-action

2.1. *The group Diff(S^1) and a family of its representations*

Let $E := C^\infty(S^1, \mathbb{R})$ be the Frechet space of smooth real-valued functions on S^1 and $F := E \otimes_{\mathbb{R}} \mathbb{C} = C^\infty(S^1, \mathbb{C})$ its complexification.

Recalling that the "differential" $\dot g$ of $g \in \mathcal{G}$ is a strictly positive, real-valued function on S^1 (see Section 1.3), we can define the following maps:

$$\tau_k : \mathcal{G} \longrightarrow \mathrm{End}\ (F)$$
$$(\tau_k(g) \cdot f)(t) := \left(\dot g\left(g^{-1}(t)\right)\right)^k f(g^{-1}(t)) ,$$

where $k \in \mathbb{C}$, $g \in \mathcal{G}$ and $f \in F$.

Lemma 14. *For all $k \in \mathbb{C}$, τ_k defines a representation of $\mathcal{G}$ by bounded linear operators on F.*

Proof. Straightforward computation. □

Remarks 15.
(1) If k is real τ_k restricts to a representation of $\mathcal{G}$ on E.
(2) F. Bien defined a bigger family of representations which depends on a second complex parameter as well, and relates it to the Verma modules of the Virasoro algebra [Bi].
(3) At least if k is an integer, these representations can be interpreted as the spaces of smooth sections of $\mathcal{G}$-homogeneous complex (resp. real) line bundles over S^1, namely $K^k = \left(TS^1 \otimes_{\mathbb{R}} \mathbb{C}\right)^k$, Of course this generalizes for arbitrary k, to the notion of the "bundle of k-densities".
(4) The mappings $\mathcal{G} \times F \longrightarrow F$, $(g, f) \longmapsto \tau_k(g) \cdot f$ are smooth mappings of Frechet manifolds (see [H] for the appropriate notion of differentiability).

For the convenience of the reader and in order to introduce relevant concepts for later use we recall the basic results on diffeomorphism groups (see e.g. [Mil] for a proof).

Proposition 16. *Let X be a compact smooth real manifold and* $\mathrm{Diff}(X)$ *the group of its diffeomorphisms. Then the following holds:*

(i) The Lie algebra of $\mathrm{Diff}(X)$ *is isomorphic to the Lie algebra* $\Xi(X)$ *of smooth vector fields on X (but with reversed commutator).*

(ii) The exponential map from $\Xi(X)$ *to* $\mathrm{Diff}(X)$ *is given by mapping a vector field ξ on X to the flow of ξ (on X) at time 1:*

$$\exp^{\mathrm{Diff}(X)} : \Xi(X) \longrightarrow \mathrm{Diff}(X),\ \xi \longmapsto \psi_1^\xi .$$

Remark 17. It should be recalled that this exponential map is not a local homeomorphism near the origin.

We are now in a position to compute easily the infinitesimal versions of the representations τ_k, which we will denote as well by τ_k. For $\xi \in \mathbf{g} := T_e \mathrm{Diff}(S^1) = \Xi(S^1)$ and ψ_s^ξ the flow of ξ on S^1, and $f \in F$ we get

$$\tau_k(\xi) \cdot f = \left.\frac{d}{dt}\right|_{s=0} (\tau_k(g_s) \cdot f) ,$$

where $g_s = \psi_s^\xi$ is interpreted as a curve in $\mathcal{G}$.

Lemma 18. *Let* $\xi = h(t)\, d/dt \in \mathbf{g} = \Xi(S^1)$ *(where* $h \in C^\infty(S^1, \mathbb{R})$*) and* $f \in F = C^\infty(S^1, \mathbb{C})$*, then*

$$(\tau_k(\xi) \cdot f)\,(t) = \left(-h(t)\dot f(t) + k\dot h(t) f(t)\right) .$$

Proof.

$$\left(\tau_k\left(h(t)\frac{d}{dt}\right) \cdot f\right)(t) = \left.\frac{d}{ds}\right|_{s=0} \left([\dot g_s(g_s^{-1}(t))]^k f(g_s^{-1}(t))\right)$$
$$= \left(\left.\frac{d}{ds}\right|_{s=0} [\dot g_s(g_s^{-1}(t))]^k f(t) + \left.\frac{d}{ds}\right|_{s=0} f(g_s^{-1}(t)\right) .$$

The second term obviously equals $-\xi(f)(t) = -h(t)\dot f(t)$. A direct calculation now shows

$$\left.\frac{d}{ds}\right|_{s=0} [\dot g_s(g_s^{-1}(t))] = \dot h(t)$$

and thus

$$\left(\tau_k\left(h(t)\frac{d}{dt}\right) \cdot f\right)(t) = k\,\dot h(t) f(t) - h(t)\dot f(t) . \qquad \square$$

2.2. The adjoint and coadjoint representation of $\mathcal{G} = \mathrm{Diff}(S^1)$

As a further application of Proposition 16 we can identify the adjoint and coadjoint representation of $\mathcal{G}$.

Lemma 19. *Let $\mathbf{g} = \Xi(S^1)$ be identified with $E = C^\infty(S^1, \mathbb{R})$ by means of the trivializing section d/dt of TS^1. Then the adjoint action of $\mathcal{G}$ on $\mathbf{g}$ identifies with τ_1 on E.*

Proof. In general the adjoint representation of a group G on its Lie algebra $\mathbf{g}$ is given by

$$\begin{aligned}\mathrm{Ad} : G \longrightarrow GL(\mathbf{g}),\ \mathrm{Ad}(g)(\xi) &= (\mathrm{int}_g)_{*_e}(\xi) \\ &= \frac{d}{ds}\Big|_{s=0} \left(g \cdot \exp^G(s\xi) g^{-1}\right) .\end{aligned}$$

Specializing to $\xi = f(t)d/dt \in \mathbf{g} = \Xi(S^1)$ with $\exp(s\xi) = \psi_s^\xi$, the flow of ξ on S^1 at time s, and $g \in \mathcal{G} = \mathrm{Diff}^+(S^1)$, we get:

$$\begin{aligned}(\mathrm{Ad}(g) \cdot \xi)\,(t) &= \frac{d}{ds}\Big|_{s=0} \left(g \circ \psi_s^\xi \circ g^{-1}(t)\right) \\ &= g_{*_{g^{-1}(t)}} \left(\xi(g^{-1}(t))\right) \\ &= \dot{g}(g^{-1}(t)) f(g^{-1}(t)) \frac{d}{dt} .\end{aligned}$$

Thus, identifying $\mathbf{g}$ with E via $\xi = f(t)\,d/dt \longmapsto f(t)$, we conclude $\mathrm{Ad} \cong \tau_1$. □

To proceed to the coadjoint representation we remark that the topological vector space $\mathbf{g}^*$ is isomorphic to $E' = (C^\infty(S^1, \mathbb{R}))'$, the space of distributions on the circle. Thus $L^1(S^1, \mathbb{R})$ is canonically embedded as a linear subspace in $\mathbf{g}^*$:

$$L^1(S^1, \mathbb{R}) \xrightarrow{\ j\ } \mathbf{g}^* = E', \quad j(\alpha)(f) := \int_0^1 f(t)\alpha(t)dt .$$

(We will suppress the letter j in the sequel.) This, in turn, yields a chain of subspaces:

$$\mathbf{g}^*_{\mathrm{reg}} := E \subset H^{k,p}(S^1, \mathbb{R}) \subset C^0(S^1, \mathbb{R}) \subset L^1(S^1, \mathbb{R}) \subset \mathbf{g}^* = E' ,$$

where $k \in \mathbb{N}^{\geq 1}$, $p \in \mathbb{N}^{\geq 1} \cup \{\infty\}$ and all inclusions are continuous embeddings relative to their respective structures as topological vector spaces.

Lemma 20. *The subspaces $\mathbf{g}^*_{\mathrm{reg}}, H^{k,p}(S^1, \mathbb{R})$, $C^0(S^1, \mathbb{R})$, and $L^1(S^1, \mathbb{R})$ are $\mathcal{G}$-invariant under the coadjoint representation of $\mathcal{G}$. This representation is isomorphic to τ_{-2} on $\mathcal{G}^*_{\mathrm{reg}} \cong E$ and is given on $L^1(S^1, \mathbb{R})$ by the obvious extension of τ_{-2}:*

$$\left(\mathrm{Ad}^*(g) \cdot \alpha\right)(t) = [\dot{g}(g^{-1}(t))]^{-2}\, \alpha(g^{-1}(t))$$

for $g \in \mathcal{G}$ and $\alpha \in L^1(S^1, \mathbb{R})$.

Proof. Let $\alpha \in \mathbf{g}^*_{\mathrm{reg}} \cong C^\infty(S^1,\mathbb{R})$, $f \in \mathbf{g} \cong C^\infty(S^1,\mathbb{R})$, and $g \in \mathcal{G}$, then:

$$\begin{aligned}(\mathrm{Ad}^*(g)\cdot\alpha)(f) &= \alpha(\mathrm{Ad}(g^{-1})\cdot f)\\ &= \int_0^1 (g^{-1})^\bullet(g(t))\, f(g(t))\alpha(t)dt\\ &= \int_0^1 [\dot{g}(g^{-1}(s))]^{-2} f(s)\alpha(g^{-1}(s))ds\\ &= \int_0^1 f(s)(\tau_{-2}(g)\cdot\alpha)(s)ds,\end{aligned}$$

thus

$$\mathrm{Ad}^*|_{\mathbf{g}^*_{\mathrm{reg}}} \cong \tau_{-2}.$$

Since the same computation applies to $\alpha \in L^1(S^1,\mathbb{R})$, the lemma is proven. □

Remark 21. The lemma clearly explains why the coadjoint vectors are often considered as "quadratic differentials": $\alpha \in \mathbf{g}^*_{\mathrm{reg}}$ transforms under $g \in \mathcal{G}$ as a section of $\otimes^2(T^*S^1)$.

We would like to describe briefly the orbit structure of $\mathcal{G}$ on $\mathbf{g}^*$ (see e.g. [Ki,Wi] for a more detailed analysis).

Proposition 22. *If $\alpha \in \mathbf{g}^*_{\mathrm{reg}}$ and α is everywhere positive, then there exists a $g \in \mathcal{G}$ such that $\mathrm{Ad}^*(g)\cdot\alpha$ is a positive constant.*

Proof. We set $I_0(\alpha) := \int_0^1 \sqrt{\alpha(t)}dt$, yielding a $\mathcal{G}$-invariant function on $\mathbf{g}^*_{\mathrm{reg},>0} := \{\alpha \in \mathbf{g}^*_{reg}|_{\alpha>0}\}$. Given $\alpha \in \mathbf{g}^*_{\mathrm{reg},>0}$ we proceed to construct a $\mathbb{R}$-valued map on S^1 by

$$g(t) := \frac{1}{I_0(\alpha)}\int_0^t \sqrt{\alpha(s)}ds,$$

which fulfills $g(0)=0$, $g(1)=1$ and $\dot{g}(t)=\sqrt{\alpha(t)}/I_0(\alpha)$. Thus g defines an element of $\mathcal{G}$—that even fixes $1 \in S^1$. We compute

$$\begin{aligned}(\mathrm{Ad}^*(g)\cdot\alpha)(t) &= [\dot{g}(g^{-1}(t))]^{-2}\alpha(g^{-1}(t))\\ &= (I_0(\alpha))^2\frac{1}{\alpha(g^{-1}(t))}\alpha(g^{-1}(t))\\ &= (I_0(\alpha))^2,\end{aligned}$$

which is a positive constant. □

Corollary 23. *The $\mathcal{G}$-invariant open cone $\mathbf{g}^*_{\mathrm{reg},>0}$ in $\mathbf{g}^*_{\mathrm{reg}}$ contains only one type of orbit, namely $\mathcal{G}/\operatorname{Rot} S^1$ and $\mathbf{g}^*_{\mathrm{reg},>0}/\mathcal{G} \cong \mathbb{R}^{>0}$. Furthermore, $\mathbf{g}^*_{\mathrm{reg},>0}$ is isomorphic to $\mathbb{R}^{>0} \times (\mathcal{G}/\operatorname{Rot} S^1)$ as $\mathcal{G}$-Frechet manifolds.*

Proof. Obviously $\mathbf{g}^*_{\mathrm{reg},>0}/\mathcal{G} \cong \mathbb{R}^{>0}$, the isomorphy given by $\alpha \longmapsto (I_0(\alpha))^2$. Let now $\alpha_0 \in \mathbb{R}^{>0}$ and $g \in \mathcal{G}$, then:

$$\begin{aligned}\alpha_0 = \mathrm{Ad}^*(g) \cdot \alpha_0 &\iff \alpha_0 = [\dot{g}(g^{-1}(t))]^{-2} \cdot \alpha_0 \\ &\iff \dot{g}(s) = 1 \quad \forall s \in [0,1] \\ &\iff g \in \operatorname{Rot} S^1,\end{aligned}$$

the subgroup of rotations of $\mathrm{Diff}(S^1)$. The last claim follows by consideration of the map

$$\mathbb{R}^{>0} \times (\mathcal{G}/S^1) \longrightarrow \mathbf{g}^*_{\mathrm{reg},>0}, \quad \chi(\alpha_0, gS^1) := \mathrm{Ad}^*(g) \cdot \alpha_0$$

and its inverse $\chi^{-1}(\alpha) = ((I_0(\alpha))^2, g(\alpha) \cdot S^1)$, where $g(\alpha)$ is constructed from α as in the proof of the above proposition. □

We would like to present explicitly a slightly technical, but later on useful "Sobolev-version" of the last proposition.

Proposition 24. *Let α be in $H^{l,2}(S^1, \mathbb{R})$ (for $l \geq 1$) and everywhere positive. Then there exists a*

$$g \in \mathcal{G}_{l+1} = \{h \in H^{l+1,2}(S^1, S^1) \mid h \text{ is bijective and } h^{-1} \in H^{l+1,2}(S^1, S^1)\}$$

with the property that

$$[\dot{g}(g^{-1}(t))]^{-2} \alpha(g^{-1}(t)) = \text{“}\mathrm{Ad}^*(g) \cdot \alpha\text{”} = (I_0(\alpha))^2.$$

Proof. Obviously, the positive number $I_0(\alpha)$ is well-defined and we can try, mimicking the proof of Proposition 22, to set

$$g(t) := \frac{1}{I_0(\alpha)} \int_0^t \sqrt{\alpha(s)}\, ds.$$

Obviously $g(0) = 0$, $g(1) = 1$ and $\dot{g}(t) = \sqrt{\alpha(t)}/I_0(\alpha) > 0$. By induction g can be shown to be in $H^{l+1,2}(S^1, S^1)$. Furthermore g is bijective and the Lebesgue integral conditions on g^{-1} follow by general considerations. Obviously "$\mathrm{Ad}^*(g) \cdot \alpha$" $= (I_0(\alpha))^2$ as in the smooth case. □

2.3. *The "moment" map of the $\mathcal{G}$-action on the loop space*

Definition 25. The "moment" map of the $\mathcal{G}$-action on X is the application $\Phi : X \longrightarrow \mathbf{g}^*$, $\gamma \longmapsto \Phi(\gamma)$, given by $\Phi(\gamma)(\xi) := -\mu_\gamma(\xi_X(\gamma))$ for $\gamma \in X, \xi \in \mathbf{g}$.

Remarks 26.

(1) If $\xi = f(t)d/dt$, then

$$\Phi(\gamma)(\xi) = -\frac{1}{2}\int_0^1 g_{\gamma(t)}(-f(t)\dot{\gamma}(t), \dot{\gamma}(t))dt$$
$$= \frac{1}{2}\int_0^1 f(t)g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))dt$$
$$= \frac{1}{2}\int_0^1 f(t)\|\dot{\gamma}\|^2 dt,$$

i.e. Φ maps a loop γ to the L^1 measure given by the energy density $\frac{1}{2}\|\dot{\gamma}\|^2$ of γ.

(2) In the case $X = X_\infty$, it follows that $\Phi(\gamma) \in \mathbf{g}^*_{\mathrm{reg}}$ for all γ. If we denote the smooth map $TM \longrightarrow \mathbb{R}$, $u \longmapsto \frac{1}{2}g(u,u)$ by q then it follows by general principles (see [H]) that the induced map

$$H_\infty(q) : C^\infty(S^1, TM) \longrightarrow C^\infty(S^1, \mathbb{R}) u \longmapsto q \circ u$$

is smooth with respect to the Frechet manifold structures. Since in this case the fundamental vector field $\zeta_X(\gamma) = -\dot{\gamma}$ is a smooth section of the tangent bundle $TX_\infty \cong C^\infty(S^1, TM)$ it follows that $\Phi = H_\infty(q) \circ (-\zeta_X)$ is smooth.

(3) This argument extends to the case $k \geq 2$ as follows: For $l \geq 1$ we can define the map $H_l(q) : H^{l,2}(S^1, TM) \cong TX_l \longrightarrow H^{l,2}(S^1, \mathbb{R})$ by $u \longmapsto q \circ u$. In analogy to the case $l = 1$ considered e.g. in [Kl] it can be shown that $H_l(q)$ is smooth, since q is smooth. Combining this with the fact that ζ_X is a smooth section in the $H^{k-1,2}$-completion $\overline{TX_k}$ of the tangent bundle of X_k and this, in turn embeds smoothly in TX_{k-1} (for $k \geq 2$) we see that $\Phi = H_{k-1}(q) \circ (-\zeta_{X_k})$ is smooth.

(4) Despite the fact that things will never really work out well in the case $k = 1$, we still sketch another proof of smoothness that works for all X:

(a) In the case that (M, g) is $\mathbb{R}^N$ with the flat euclidean metric g_0 the moment map reads

$$\Phi_0(\gamma) = \tfrac{1}{2}\|\dot{\gamma}\|^2.$$

This is the composition of the linear map

$$H^{k,2}(S^1, \mathbb{R}^n) \longrightarrow H^{k-1,2}(S^1, \mathbb{R}^N), \ \gamma \longmapsto \dot{\gamma}$$

with the bilinear map $(u, v) \longmapsto u \cdot v = \sum_{j=1}^N u^j \cdot v^j$. This yields that in the case $k = 1$

$$\Phi_0 : H^{1,2}(S^1, \mathbb{R}^N) \longrightarrow L^1(S^1, \mathbb{R}^N)$$

is smooth and in the case $k \geq 2$

$$\Phi_0 : H^{k,2}(S^1, \mathbb{R}^N) \longrightarrow H^{k-1,2}(S^1, \mathbb{R}^N)$$

is smooth. (Note that $H^{p,2}(S^1, \mathbb{R}^N)$ is a Hilbert algebra for $p \geq 1$ (!), but not for $H^{0,2} = L^2$).

(b) By the Nash Embedding Theorem (see e.g. [Gr]) there exists a smooth isometric embedding $f : (M, g) \longrightarrow (\mathbb{R}^N, g_0)$ for N sufficiently big. This f induces a map $H_k(f) : H^{k,2}(S^1, M) \longrightarrow H^{k,2}(S^1, \mathbb{R}^N), \gamma \longmapsto f \circ \gamma$, which is smooth since its differential is $H_k(f_*) : H^{k,2}(S^1, TM) \longrightarrow H^{1,2}(S^1, T\mathbb{R}^N)$ (see again [Kl]). The moment map of X_k is now realized as $\Phi = \Phi_0 \circ H_k(f)$ and thus smooth.

Lemma 27. *The moment map $\Phi : X \longrightarrow \mathbf{g}^*$ is $\mathcal{G}$-equivariant.*

Proof. Since $\Phi(X)$ lies in $L^1(S^1, \mathbb{R})$ in all cases, the action on $\mathbf{g}^*$ restricts to τ_{-2}. The result follows now by straightforward computation. □

2.4. Summary of the "classical momentum geometry"

Before we proceed in Section 3 to analyse special properties of the above defined map, we collect some elementary facts about G-equivariant moment maps to have an idea what we could ideally expect to hold:

Proposition 28. *Let G be a finite dimensional Lie group with Lie algebra $\mathbf{g}$ acting smoothly on a finite dimensional symplectic manifold (X, ω) and $\Phi : X \longrightarrow \mathbf{g}^*$ a map fulfilling*

(i) Φ is smooth and G-equivariant.

(ii) For every ξ in $\mathbf{g}$ the function $\xi \circ \Phi =: \Phi^\xi$ is a Hamiltonian for the fundamental vector field ξ_X on X, i.e. $\omega_x(\xi_X(x), u) = d\Phi^\xi(u)$ for all $u \in T_xX$.

Then Φ is called "G-equivariant moment map for the G-action on X" and the following holds:

(1) $(D\Phi)_x : T_xX \longrightarrow T_{\Phi(x)}\mathbf{g}^* \cong \mathbf{g}^*$ is dual to the map

$$\sigma_x : \mathbf{g} \longrightarrow (T_xX)^*, \ \xi \longmapsto \omega_x(\xi_X(x), \bullet) .$$

(2) $\mathrm{Im}(D\Phi)_x = \mathrm{Ann}(\mathbf{g}_x) \subset \mathbf{g}^*$, where $\mathbf{g}_x$ is the Lie algebra of the stabilizer G_x of x, and

$$\mathrm{Ker}(D\Phi)_x = (T_xG \cdot x)^{\angle} = \{v \in T_xX \mid \omega(v, u) = 0 \quad \forall u \in T_xG \cdot x\} ,$$

the symplectic perpendicular to the tangent space of the G-orbit through x.

(3) It follows especially that x is a regular point of Φ iff G_x is discrete and in this case

$$T_x(\Phi^{-1}(\Phi(x)) = (T_xG \cdot x)^{\angle} .$$

(4) Denoting the inclusion of the G-orbit $G \cdot x$ (for a fixed $x \in X$) by $i : G \cdot x \longrightarrow X$, we find

$$i^* \circ \Phi^* \omega_{\mathcal{O}} = i^* \omega ,$$

where $\omega_{\mathcal{O}}$ is the canonical symplectic form on the coadjoint orbit $\mathcal{O} = G \cdot \Phi(x)$.

Proof. Folklore (see e.g. [Gu-St]). □

We remark that this proposition holds also for suitable strong notions of symplectic manifolds and group actions in the category of Banach manifolds and Lie groups (see e.g. [Ab-Ma]).

3. Properties of the "moment" map Φ

3.1. Image and fibres of Φ

Before going into a detailed study of the infinitesimal properties of Φ, we will analyse the first global properties of this map. We prepare ourselves with an easy topological observation.

Lemma 29.
(i) *The set $\{\alpha \in C^\infty(S^1, \mathbb{R}) \mid \alpha > 0\}$ is open in $C^\infty(S^1, \mathbb{R})$ and its closure is the set of non-negative smooth functions.*
(ii) *For $k \geq 2$, the set $\{\alpha \in H^{k,2}(S^1, \mathbb{R}) \mid \alpha > 0\}$ is well-defined and open, its closure being $\{\alpha \in H^{k,2}(S^1, \mathbb{R}) \mid \alpha \geq 0\}$.*
(iii) *The set $\{\alpha \in L^1(S^1, \mathbb{R}) \mid \alpha \geq 0$ almost everywhere $\}$ has empty interior with respect to the L^1-topology.*

Proof. The first two assertions follow from the analogous statement for continuous functions and the fact that the considered spaces embed continuously into $C^0(S^1, \mathbb{R})$.
The third claim follows from elementary properties of the Lebesgue-integral. □

Proposition 30. *Let $\Phi : X_\infty \longrightarrow \mathfrak{g}^*_{\mathrm{reg}} \cong C^\infty(S^1, \mathbb{R})$ be the moment map on the space of smooth loops. Then the following holds:*
(i) $\{\alpha \geq 0\} \supset \Phi(X_\infty) \supset \{\alpha > 0\}$.
(ii) $\{\alpha > 0\}$ *is relatively open and dense in* $\Phi(X_\infty)$.

Proof. The left inclusion of (i) is evident. We recall now that, given a smooth, strictly positive function α on the circle, we find an element g of $\mathcal{G}$ such that $\mathrm{Ad}^*(g) \cdot \alpha = (I_0(\alpha))^2$, the constant given by $I_0(\alpha) = \int_0^1 \sqrt{\alpha(t)}dt$ (see Proposition 22 in Section 2.2).

Since $I_0(\Phi(\gamma)) = 2^{-1/2} \int_0^1 \|\dot{\gamma}(t)\| dt$ is, up to a constant factor, the length of γ, it suffices to find a smooth immersive, arclength parametrized curve γ of length $\sqrt{2} I_0(\alpha)$.

By considering "planar circles" in a coordinate chart of M and covering them multiply—if necessary—it follows that the "length spectrum" of the set of immersive smooth (even nullhomotopic) curves in M is $\mathbb{R}^{>0}$. This yields the right inclusion. Part (ii) follows by the preceding lemma. □

Analogous results hold for Sobolev loops:

Proposition 31. *Let $\Phi : X_k \longrightarrow H^{k-1,2}(S^1,\mathbb{R})$ $(k \geq 2)$ and $\Phi : X_1 \longrightarrow L^1(S^1,\mathbb{R})$ $(k = 1)$ be the moment map. Then the following holds:*

(i) $\{\alpha \in L^1(S^1,\mathbb{R}) \mid \alpha \geq 0 \text{ a. e.}\} = \Phi(X_1)$.

(ii) $\{\alpha \in H^{k-1,2}(S^1,\mathbb{R}) \mid \alpha \geq 0\} \supset \Phi(X_k) \supset \{\alpha > 0\}$ *for* $k \geq 2$.

(iii) The set $\{\alpha > 0\}$ *is relatively open and dense in* $\Phi(X_k)$ *for* $k \geq 2$.

Proof. We begin with the case $k \geq 2$. By Proposition 24 in Section 2.2 we find, given $\alpha > 0$, a $H^{k,2}$-diffeomorphism g of the circle with the property $\mathrm{Ad}^*(g) \cdot \alpha = (I_0(\alpha))^2$. Taking a smooth loop γ with $\Phi(\gamma) = (I_0(\alpha))^2$ we can solve the problem by considering the $H^{k,2}$-loop $\gamma \circ g$. Thus (ii) is proven and (iii) is again a direct consequence of Lemma 29.

In the case $k = 1$ we follow again the same approach. Given $\alpha \geq 0$ almost everywhere, the definition

$$g(t) := \frac{1}{I_0(\alpha)} \int_0^t \sqrt{\alpha(s)}ds$$

yields an element of $H^{1,2}([0,1],\mathbb{R})$ fulfilling $g(0) = 0$ and $g(1) = 1$. Taking g as a map from S^1 to S^1 and denoting with γ a smooth loop with the property $\Phi(\gamma) = (I_0(\alpha))^2$, the loop $\gamma \circ g$ is in X_1 and is mapped by Φ onto α. □

Turning our attention to the fibres of Φ, the "generic" case can easily be settled.

Proposition 32. *Let* $\Phi : X_\infty \longrightarrow \mathbf{g}^*_{\mathrm{reg}}$. *Then*

(i) The Φ*-fibre over a positive constant* $\alpha_0 > 0$ *consists of the set of smooth arclength parametrized loops with length* $\sqrt{2\alpha_0}$ *or equivalently energy* α_0.

(ii) The Φ*-fibre over an arbitrary, positive* $\alpha \in \mathbf{g}^*_{\mathrm{reg}}$ *is diffeomorphically mapped onto the fibre over* $(I_0(\alpha))^2$ *by* ϑ_g*, where* g *is an appropriate element of* $\mathcal{G}$.

Proof. The first statement is evident since $\Phi(\gamma) = \frac{1}{2}\|\dot{\gamma}\|^2$ and the second follows by taking g as in the proof of Proposition 30 above. □

Remarks 33.

(1) Analogous statements hold for X_k in the case $k \geq 2$. To this end, one has only to observe that the $H^{k,2}$-diffeomorphism g constructed in the proof of Proposition 31 gives rise to a diffeomorphism "$\mathrm{Ad}^*(g)$" of $H^{k-1,2}(S^1,\mathbb{R})$ and a diffeomorphism "ϑ_g" of X_k and these two are connected by Φ:

$$\mathrm{Ad}^*(g) \circ \Phi = \Phi \circ \vartheta_g .$$

(2) For $k = 1$, the nice characterization of the fibres obviously breaks down.

3.2. Image and kernel of $D\Phi$

We compute the differential of Φ for X_∞ and X_k simultaneously.

Proposition 34. *Let $\Phi : X \longrightarrow \mathbf{g}^*$ be the moment map, $\gamma \in X$ and $u \in T_\gamma X$. Then we have:*

$$(D\Phi)_\gamma(u)(t) = g_{\gamma(t)}\left(\frac{\nabla u}{dt}(t), \dot\gamma(t)\right),$$

where the RHS is in $\mathbf{g}^*_{\mathrm{reg}} = C^\infty(S^1,\mathbb{R})$ for X_∞, in $H^{k-1,2}(S^1,\mathbb{R})$ for X_k with $k \geq 2$ and in $L^1(S^1,\mathbb{R})$ for X_1.

Proof. Let $\xi = f(t)d/dt$ be in $\mathbf{g} = C^\infty(S^1,\mathbb{R})$ and ξ_X the corresponding "fundamental vector field" on X. Defining a smooth curve γ_s in X by

$$\gamma_s(t) := \exp^M_{\gamma(t)}(s \cdot u(t))$$

we can calculate as follows:

$$\begin{aligned}
(D\Phi)_\gamma(u)(\xi) &= d(\xi \circ \Phi)_{\gamma(u)} \\
&= -d(\mu(\xi_X))_\gamma(u) \\
&= -u(\mu(\xi_X)) \\
&= \frac{d}{ds}\bigg|_{s=0} \frac{1}{2}\int_0^1 g_{\gamma_s(t)}\left(f(t)\dot\gamma_s(t), \dot\gamma_s(t)\right) dt \\
&= \frac{1}{2}\int_0^1 f(t)\left[\frac{d}{ds} g_{\gamma_s(t)}\left(\dot\gamma_s(t), \dot\gamma_s(t)\right)\right]\bigg|_{s=0} dt \\
&= \frac{1}{2}\int_0^1 f(t)\left[2 g_{\gamma_s(t)}\left(\frac{\nabla}{ds}\dot\gamma_s(t), \dot\gamma_s(t)\right)\right]\bigg|_{s=0} dt .
\end{aligned}$$

Interpreting γ_s as a map

$$\Gamma : (-\varepsilon,\varepsilon) \times S^1 \longrightarrow M,\ \Gamma(s,t) := \gamma_s(t)$$

we find

$$\frac{\nabla}{ds}\dot\gamma_s(t) = \nabla_{d/ds}\Gamma_*\left(\frac{d}{dt}\right) = \nabla_{d/dt}\Gamma_*\left(\frac{d}{ds}\right) + \Gamma_*\left(\left[\frac{d}{ds}, \frac{d}{dt}\right]\right)$$

(see e.g. [Ga-Hu-La, p. 144] for the last equation) and thus

$$\frac{\nabla}{ds}\dot\gamma_s(t) = \frac{\nabla}{dt}\left(\frac{d}{ds}\gamma_s\right)$$

and this equals ∇/dt for $s = 0$. Inserting this last result we find

$$(D\Phi)_\gamma(u)(\xi) = \int_0^1 f(t) g_{\gamma(t)}\left(\frac{\nabla u}{dt}(t), \dot\gamma(t)\right) dt,$$

i.e.

$$(D\Phi)_\gamma(u)(t) = g_{\gamma(t)}\left(\frac{\nabla u}{dt}, \dot\gamma(t)\right). \qquad \square$$

Corollary 35. *Let $\gamma \in X$, $u \in T_\gamma X$ and $\xi = f(t)d/dt \in \mathfrak{g}$, then*

$$d(\xi \circ \Phi)_\gamma(u) = -\check\omega(u, \xi_X(\gamma)),$$

i.e. the smooth function $\Phi^\xi := \xi \circ \Phi$ is the "Hamiltonian" for the field ξ_X (in the sense of Proposition 28)—despite the fact that ω has degeneration and the $\mathcal{G}$-action on X is not differentiable.

Proof. Recalling the definition

$$\check\omega_\gamma(u, v) = \int_0^1 g_{\gamma(t)}\left(\frac{\nabla u}{dt}(t), v(t)\right) dt$$

for $u \in \overline{T_\gamma X}$, the $H^{k-1,2}$-completion of $T_\gamma X$ and $v \in L(X)$, the L^2-completion of $T_\gamma X$ and the fact $\xi_X(\gamma)(t) = -f(t)\dot\gamma(t)$ the result follows immediately from the proposition. $\square$

For γ in X we denote the "tangent space to the $\mathcal{G}$-orbit through γ" by

$$V_\gamma := \{f(t)\dot\gamma(t) \mid f \in C^\infty(S^1(\mathbb{R})\}$$

and remark that $V_\gamma \subset T_\gamma X_\infty$ for $\gamma \in X_\infty$ and $V_\gamma \subset \overline{T_\gamma X}$, the $H^{k-1,2}$-closure, for $\gamma \in X_k$.

Corollary 36. *Let γ be in X, then*

$$\ker(D\Phi)_\gamma = (V_\gamma)^\angle := \{u \in T_\gamma X \mid \omega(u, v) = 0 \quad \forall v \in V_\gamma\}.$$

Proof. Let $u \in \ker(D\Phi)_\gamma$ and $v = f(t)\dot\gamma(t) \in V_\gamma$. Then

$$\begin{aligned}\check\omega(u, v) &= \int_0^1 g_{\gamma(t)}\left(\frac{\nabla u}{dt}(t), f(t)\dot\gamma(t)\right) dt \\ &= \int_0^1 f(t)\left[g_{\gamma(t)}\left(\frac{\nabla u}{dt}(t), \dot\gamma(t)\right)\right] dt \\ &= \int_0^1 f(t)\left[(D\Phi)_\gamma(u)(t)\right] dt.\end{aligned}$$

Now the claim follows immediately. □

We remark that the preceding two corollaries indicate that Φ behaves pretty much as a classical moment map. This is further underlined by the following observation and will be confirmed by the consideration of the image of $(D\Phi)_\gamma$ below.

Lemma 37. *Let $\Phi : X_\infty \longrightarrow \mathbf{g}^*_{\mathrm{reg}}$ and γ be a smooth loop, then*

$$i^* \circ \Phi^* \omega_{\mathcal{O}} = i^* \omega ,$$

where $i : \mathcal{G} \cdot \gamma \longrightarrow X_\infty$ is the inclusion of the orbit and $\omega_{\mathcal{O}}$ is the canonical symplectic structure on $\mathcal{O} = \mathcal{G} \cdot \Phi(\gamma)$.

Proof. Let $\xi = f(t)d/dt, \eta = h(t)d/dt$ be in $\mathbf{g}$ and recall $[\xi,\eta]_{\mathbf{g}} = (f\dot{h} - \dot{f}h)d/dt$. Thus the canonical symplectic structure on $\mathcal{O}$ is given by

$$\begin{aligned}\omega_{\mathcal{O}}(\xi,\eta)(\Phi(\gamma)) &= \Phi(\gamma)([\xi,\eta]) \\ &= \frac{1}{2}\int_0^1 (f\dot{h} - \dot{f}h)\|\dot{\gamma}(t)\|^2 dt,\end{aligned}$$

where $\|\dot{\gamma}(t)\|^2$ is used for $g_{\gamma(t)}(\dot{\gamma}(t),\dot{\gamma}(t))$. The 2-form ω of X_∞ restricted to the orbit $\mathcal{G} \cdot \gamma$ reads as

$$\begin{aligned}\omega(\xi_X,\eta_X)(\gamma) &= \int_0^1 g_{\gamma(t)}\left(-f\dot{\gamma}, -\frac{\nabla}{dt}(h\dot{\gamma})\right) dt \cdot \\ &= \int_0^1 \left\{ f\dot{h}\|\dot{\gamma}\|^2 + fh\, g_{\gamma(t)}\left(\dot{\gamma}, \frac{\nabla\dot{\gamma}}{dt}\right)\right\} dt \\ &= \int_0^1 \left\{ f\dot{h}\|\dot{\gamma}\|^2 + \frac{fh}{2}\frac{d}{dt}\left(\|\dot{\gamma}\|^2\right)\right\} dt \\ &= \frac{1}{2}\int_0^1 \{f\dot{h} \cdot 2 - \dot{f}h - f\dot{h}\}\,\|\dot{\gamma}\|^2 dt \\ &= \omega_{\mathcal{O}}(\xi,\eta)(\Phi(\gamma)) .\end{aligned}$$

□

In order to find the image of the differential $D\Phi$ we recall the construction of the "développement de Cartan" (see [Ca]):

Given a curve $\gamma : [0,1] = I \longrightarrow M$, let $P_\gamma|_t^s$ denote the parallel transport along γ, which is an isometry between $T_{\gamma(t)}M$ and $T_{\gamma(s)}M$. For the sake of brevity we denote $P_\gamma|_0^t$ by P_t in the sequel. We call the curve of "velocities" $P_t^{-1}(\dot{\gamma}(t))$ in $T_{\gamma(0)}M$ $y(t)$ and set $x(t) := \int_0^t y(s)ds$. This curve x in the euclidean space $(T_{\gamma(0)}M, g_{\gamma(0)})$ is called the "développement de Cartan". We can establish an isometric isomorphism between the vector bundle γ^*TM with the metric γ^*g and the trivial $T_{\gamma(0)}M$-bundle over I with the constant metric $g_{\gamma(0)}$ by mapping $u_t \in (\gamma^*TM)_t = T_{\gamma(t)}M$ to $P_t^{-1}(u_t) \in T_{\gamma(0)}M$. This induces an isomorphism

$$\Gamma(I, \gamma^*TM) \longrightarrow \text{Map}\,(I, T_{\gamma(0)}M),\; u \longmapsto \mathsf{u}\,,$$

where $\mathsf{u}(t) := P_t^{-1}(u(t))$, with the property that the covariant derivative ∇/dt on the former space is identified with the ordinary derivative d/dt on the latter.

This yields already the notable result that γ is a geodesic iff x is a piece of a ray through 0.

In case γ is closed we have a distinguished operator $P_1 : T_{\gamma(0)}M \longrightarrow T_{\gamma(0)}M$ and the condition $u(0) = u(1)$ for $u \in \Gamma(I, \gamma^*TM)$ reads as $P_1(\mathsf{u}(1)) = \mathsf{u}(0)$ for $\mathsf{u} \in \text{Map}(I, T_{\gamma(0)}M)$.

For a loop $\gamma \in X$ and $u \in T_\gamma X = \{u \in \Gamma(I, \gamma^*TM) \mid u(0) = u(1)\}$ the differential of Φ was in Proposition 34 calculated as

$$(D\Phi)_\gamma(u)(t) = g_{\gamma(t)}\left(\frac{\nabla u}{dt}(t),\, \dot\gamma(t)\right)\,,$$

yielding the equivalent expression $\dot x(t)\dot{\mathsf{u}}(t)$, for x being the développement de Cartan, u in $\text{Map}(I, T_{\gamma(0)}M)$ such that $P_1(\mathsf{u}(1)) = \mathsf{u}(0)$ and "$\cdot$" denoting the scalar product $g_{\gamma(0)}$ on $T_{\gamma(0)}M$.

Proposition 38. *Let γ be an immersive element of X_∞. Then the differential of Φ in γ*
(i) is surjective onto $\mathbf{g}^*_{\text{reg}}$ if γ is not *a geodesic,*
(ii) has a one-codimensional image if γ is a geodesic.

Proof. Ad (i) Given $\alpha \in C^\infty(S^1, \mathbb{R})$ we seek $\mathsf{u} \in C^\infty(I, T_{\gamma(0)}M)$ such that $\dot x \cdot \dot{\mathsf{u}} = \alpha$ and $P_1(\mathsf{u}(1)) = \mathsf{u}(0)$. The Ansatz

$$v(t) := \int_0^t \left(\frac{\dot x \alpha}{\|\dot x\|^2}\right)(s)\,ds$$

reduces us to the following problem $(*)$:

Given $\mathsf{v}_1 := \int_0^1 \left(\dot x\alpha/\|\dot x\|^2\right)(s)\,ds$, find $\mathsf{w} \in C^\infty(I, T_{\gamma(0)}M)$ such that $\dot x \cdot \dot{\mathsf{w}} = 0$ and $\mathsf{w}(1) - P_1^{-1}(\mathsf{w}(0)) = \mathsf{v}_1$.

Having solved $(*)$, the map $\mathsf{u} := v - w$ fulfills

$$\dot x \cdot \dot{\mathsf{u}} = \dot x \cdot \dot{\mathsf{v}} = \alpha$$

and

$$P_1(\mathsf{u}(1)) = P_1(\mathsf{v}(1)) - P_1(\mathsf{w}(1)) = -\mathsf{w}(0) = \mathsf{u}(0)\,,$$

i.e. $(D\Phi)_\gamma(u) = \alpha$.

In order to solve $(*)$ we consider the vector space $H := \{h : I \longrightarrow T_{\gamma(0)}M \mid h$ is smooth, $h(0) = 0$ and $h(z) \perp \dot{x}(t)\ \forall t\}$ and the linear map $J : H \longrightarrow T_{\gamma(0)}M$, $J(h) := \int_0^1 h(s)ds$.

We make the following claim:

J is surjective. $(**)$

To prove $(**)$ we observe that since γ is no geodesic, x is not a straight line and we thus can find two points $t_1, t_2 \in (0,1)$ such that $(\dot{x}(t_1)^{\perp}) + (\dot{x}(t_2))^{\perp} = T_{\gamma(0)}M$. Given now a point v_1 in $(\dot{x}(t_1)^{\perp})$, the $g_{\gamma(0)}$ perpendicular to $\dot{x}(t_1)$ in $T_{\gamma(0)}M$, we find an element $h \in H$ that approximates "v_1 times the δ-measure in t_1" arbitrarily good as a measure. Thus $\|J(h) - v_1\|$ can be made arbitrarily small. Since J is linear its image is a linear (and automatically closed) subspace of $T_{\gamma(0)}M$ and therefore the above argument implies that $\dot{x}(t_1)^{\perp}$ (and by the same argument $\dot{x}(t_2)^{\perp}$) is contained in $\operatorname{Im} J$, yielding $(**)$.

Now, taking $h \in H$ such that $J(h) = v_1$, v_1 as in problem $(*)$, we define

$$\mathrm{w}(t) := \int_0^t h(s)ds$$

and find $\dot{x} \cdot \dot{\mathrm{w}} = 0$ as well as $\mathrm{w}(1) = \mathrm{v}_1$, $\mathrm{w}(0) = 0$ thus solving $(*)$.

Ad (ii) We claim that in this case

$$\operatorname{Im}(D\Phi)_\gamma = \{\alpha \in C^\infty(S^1, \mathbb{R}) \mid \textstyle\int_0^1 \alpha(s)ds = 0\}.$$

For the sufficiency of the condition on the mean we note that—defining the constant (!) vector $n := \dot{x}$, the velocity of x—the first Ansatz in (i)

$$\mathrm{u}(t) := \left(\int_0^t \alpha(s)ds\right) \frac{n}{\|n\|^2}$$

already solves the problem $\dot{x} \cdot \dot{\mathrm{u}} = \alpha$, $P_1(\mathrm{u}(1)) = \mathrm{u}(0)$.

Assuming now that $(*)\dot{x} \cdot \mathrm{u} = \alpha$ for $\mathrm{u} \in C^\infty(I, T_{\gamma(0)}M)$ such that $P_1(\mathrm{u}(1)) = \mathrm{u}(0)$, we denote by $\mathrm{u}^{\|}(t)$ and $\mathrm{u}^{\perp}(t)$ the parallel and perpendicular part, respectively, of u with respect to n. Since γ is a geodesic $\dot{\gamma}$ is parallel and especially $P_1(\dot{\gamma}(0)) = \dot{\gamma}(1) = \dot{\gamma}(0)$. Therefore P_1 respects the decomposition in parallel and perpendicular parts of any vector in $T_{\gamma(0)}M$.

This gives the equations

$$P_1(\mathrm{u}^{\|}(1)) = \mathrm{u}^{\|}(0), \quad P_1(\mathrm{u}^{\perp}(1)) = \mathrm{u}^{\perp}(0),$$

showing that $\mathrm{u}^{\|}$ already solves $(*)$. Replacing u by $\mathrm{u}^{\|}$ and representing it as $\beta(t)n$ one finds $\dot{\beta}(t) = \alpha(t)/\|n\|^2$ and thus

$$\mathrm{u}(t) = \left(\int_0^t \alpha(s)ds\right) \frac{n}{\|n\|^2} + \beta(0)n,$$

$$\mathrm{u}(0) = \beta(0)n \quad \text{and} \quad \mathrm{u}(1) = \left(\int_0^1 \alpha(s)ds \right) \frac{n}{\|n\|^2} + \beta(0)n .$$

The boundary condition $P_1(\mathrm{u}(1)) = \mathrm{u}(0)$ is now equivalent to $\int_0^1 \alpha(s)ds = 0$, since $P_1(n) = n$. □

Remarks 39.

(1) We remark that the same lines of proof give the same results for the moment map

$$\Phi : X_k \longrightarrow H^{k-1,2}(S^1, \mathbb{R}) \quad \text{for } k \geq 2 ,$$

since $H^{l,2}(S^1, \mathbb{R})$ is a Hilbert algebra for $l \geq 1$. Again the case $k = 1$ is not well-behaved.

(2) Unfortunately propositions 31 and 32 in Section 3.1 as well as proposition 38 above give only sufficient information about the "generic case", i.e. immersive loops. The behaviour in general, notably the description of the set $\Phi(X) \cap$ (boundary of $\{\alpha \geq 0\}$) seems to be as interesting as difficult to access.

(3) We would like to stress the close analogy to the classical momentum geometry we have encountered so far (Corollaries 35 and 36, Lemma 37); most notably the following one: Since for immersive γ the isotropy subgroup in $\mathcal{G}$ is finite, the isotropy subalgebra $\mathbf{g}_\gamma$ is trivial. This suggests by Proposition 28 the surjectivity of the differential, which is indeed a fact for non-geodesic curves and violated by codimension one for geodesics. This latter gap can be explained by the observation that γ is geodesic iff $\dot{\gamma}$ is in kern ω_γ (see below for finer aspects of the infinitesimal relations between Φ-fibres and the orbits of $\mathcal{G}$).

3.3. The relation between $\mathcal{G}$-orbits and the "moment" fibres. Symplectic reduction

To simplify statements and calculations in this section, we introduce (recall) the following (bunch of) notations: Let γ be a loop, then

$$\begin{aligned}
\mathcal{G} \cdot \gamma &= \{ g \cdot \gamma \mid g \in \mathcal{G} \}, \\
\mathcal{G}_\gamma &= \{ g \in \mathcal{G} \mid g \cdot \gamma = \gamma \} , \\
\mathcal{O}_\gamma &:= \mathcal{G} \cdot \Phi(\gamma), \\
\Phi_\gamma &:= \Phi^{-1}(\Phi(\gamma)) , \\
V_\gamma &= \{ f\dot{\gamma} \mid f \in C^\infty(S^1, \mathbb{R}) \} \subset T_\gamma X_\infty \text{ resp. } \overline{T_\gamma X_k}
\end{aligned}$$

(the closure with respect to $H^{k-1,2}$-topology),

$$\begin{aligned}
N_\gamma &= \operatorname{kern} \omega_\gamma \subset T_\gamma X , \\
K_\gamma &:= \operatorname{kern}(D\Phi)_\gamma \subset T_\gamma X , \\
R_\gamma &:= ((\dot{\gamma}))_{\mathbb{R}} \subset V_\gamma .
\end{aligned}$$

Lemma 40. *Let γ be an immersive element of X_∞, then $\mathcal{G}\cdot\gamma\cap\Phi_\gamma\simeq S^1$.*

Proof. Let g be in $\mathcal{G}$ such that $g\cdot\gamma$ is parametrized by arclength ("p.a.l." in the sequel). Then $\mathcal{G}\cdot\gamma=\mathcal{G}\cdot(g\cdot\gamma)$ and Φ_γ is diffeomorphic to $\vartheta_g(\Phi_\gamma)=\Phi_{g\cdot\gamma}$, thus we are reduced to the case that γ is p.a.l. and hence $\Phi(\gamma)=\alpha_0$ is a positive constant and

$$\Phi_\gamma=\{\gamma\in X_\infty \mid \|\dot\gamma(t)\|^2=2\alpha_0\quad\forall t\}.$$

The equation

$$\|(\gamma\circ g)^\bullet(t)\|^2=\|\dot\gamma(g(t))\|^2\cdot|\dot g(t)|^2$$

along with the fact that $|\dot g(t)|^2\equiv 1$ iff $g\in\operatorname{Rot}S^1$ implies

$$\mathcal{G}\cdot\gamma\cap\Phi_\gamma=(\operatorname{Rot}S^1)\cdot\gamma.$$

The last RHS is a circle, since γ is non-constant. □

We remark that, again working with appropriate Sobolev diffeomorphisms, the lemma holds for $k\geq 2$.

Proposition 41. *Let γ be an immersive element of X_∞, parametrized by arclength. Then the following holds:*

(i) $V_\gamma\cap K_\gamma=R_\gamma$.

(ii) If γ is a closed geodesic, $V_\gamma+V_\gamma^\angle=T_\gamma X_\infty$.

(iii) If γ is not *a closed geodesic,*

$$(V_\gamma+V_\gamma^\angle)\oplus((J\dot\gamma))_\mathbb{R}=T_\gamma X_\infty.$$

Furthermore, in this case, we can find a one-dimensional, locally closed submanifold $S_\gamma\subset X_\infty$ such that

$$(V_\gamma+V_\gamma^\angle)\oplus T_\gamma S_\gamma=T_\gamma X_\infty.$$

(iv) $K_\gamma^\angle=V_\gamma+N_\gamma$, $K_\gamma\cap K_\gamma^\angle=R_\gamma+N_\gamma$.

Proof. For the proof we introduce further shorthands:

$$\alpha_0:=\Phi(\gamma)=\tfrac{1}{2}\|\dot\gamma\|^2,$$
$$u\cdot v:=g_{\gamma(t)}(u(t),v(t)),$$

for $u,v\in T_\gamma X_\infty$ and we denote $\nabla u/dt$ by $\dot u$, e.g. $\ddot\gamma=\nabla\dot\gamma/dt$.

Ad (i) Recall that

$$V_\gamma\cap K_\gamma=\left\{f\dot\gamma\;\middle|\; f\in C^\infty(S^1,\mathbb{R})\text{ and }\dot\gamma\cdot\left(\frac{\nabla}{dt}f\dot\gamma\right)=0\quad\forall t\right\}.$$

Since γ is p.a.l. we have $\dot\gamma(t)\cdot\ddot\gamma=0$ throughout and thus

$$\dot\gamma\cdot\left(\frac{\nabla}{dt}f\dot\gamma\right)=\dot f\|\dot\gamma\|^2+f\dot\gamma\cdot\ddot\gamma=2\dot f\alpha_0.$$

Thus $f\dot{\gamma}$ is in $K_\gamma = \mathrm{Kern}(D\Phi)_\gamma$ iff f is constant.

Ad (ii) For γ immersive, the bundle γ^*TM is the direct orthogonal sum of two smooth subbundles

$$\gamma^*TM = ((\dot{\gamma})) \oplus ((\dot{\gamma}))^\perp ,$$

where

$$((\dot{\gamma}))_t := ((\dot{\gamma}(t)))_{\mathbb{R}} \subset (\gamma^*TM)_t = T_{\gamma(t)}M$$

and the orthogonal complement is taken with respect to the metric γ^*g on γ^*TM. This yields a decomposition into closed (!) subspace:

$$\Gamma_{C^\infty}(S^1, \gamma^*TM) = \Gamma_{C^\infty}\left(S^1, ((\dot{\gamma}))\right) \oplus \Gamma_{C^\infty}\left(S^1, ((\dot{\gamma}))^\perp\right) .$$

Given now γ a closed geodesic and $w \in T_\gamma X_\infty$, we decompose the latter according to the above direct sum $w = a\dot{\gamma} + u$, where $a \in C^\infty(S^1, \mathbb{R})$ and $u(t) \cdot \dot{\gamma}(t) = 0$ for all t. Since $a\dot{\gamma}$ is in V_γ it is sufficient to show that u lies in $V_\gamma^\angle$:

$$\omega(u, f\dot{\gamma}) = -\int_0^1 \{\dot{f}(u \cdot \dot{\gamma}) + f(u \cdot \ddot{\gamma})\}dt = 0 ,$$

since $u \cdot \dot{\gamma} = 0$ and $\ddot{\gamma} = 0$, γ being a geodesic.

Ad (iii) Let us recall that $N_\gamma = \mathrm{Kern}\,\omega_\gamma = \mathrm{Kern}(\nabla/dt)$ (see Lemma 11 in Section 1.4) and that γ is thus geodesic iff $\dot{\gamma} \in N_\gamma$. Furthermore we denote the operator J_γ by J and recall $\omega_\gamma(u, Ju) > 0$ for all $u \in T_\gamma X_\infty \setminus N_\gamma$ (see Lemma 13). Thus in the case at hand we have ($\omega := \omega_\gamma$ here)

$$\omega(\dot{\gamma}, J\dot{\gamma}) > 0 .$$

The single linear equation

$$\omega(\dot{\gamma}, w) = 0 \qquad (\text{for } w \in T_\gamma X_\infty)$$

defines a one-codimensional closed subspace in $T_\gamma X_\infty$. It follows that

$$T_\gamma X_\infty = (\dot{\gamma})^\angle \oplus ((J\dot{\gamma}))_{\mathbb{R}} .$$

Our strategy is now simply to show the following, which is obviously sufficient:

$$(\dot{\gamma})^\angle = V_\gamma + V_\gamma^\angle . \qquad (*)$$

To prove the inclusion of the RHS in the LHS, note that obviously $V_\gamma^\angle \subset (\dot{\gamma})^\angle$.

Let now $f\dot{\gamma}$ be in V_γ, then

$$\omega(\dot{\gamma}, f\dot{\gamma}) = -\int_0^1 \ddot{\gamma} \cdot f\dot{\gamma} dt = 0 ,$$

since γ is p.a.l. Thus V_γ is contained in $(\dot\gamma)^\angle$ as well. To show the inclusion of the LHS in the RHS, let w in $(\dot\gamma)^\angle$ be decomposed as

$$w = a\dot\gamma + u\,,$$

where $a \in C^\infty(S^1,\mathbb{R})$ and $u(t)\cdot\dot\gamma(t) = 0$ for all t.

Since w is in $(\dot\gamma)^\angle$, the mean value of the function $u\cdot\ddot\gamma$ is zero. Thus the function $b(t) := (1/2\alpha_0)\int_0^t u(s)\cdot\ddot\gamma(s)ds$ is a smooth function on the circle ($\simeq [0,1]/\sim$). We define $v := u + b\dot\gamma$ and claim

$$v \in V_\gamma^\angle\,. \tag{**}$$

Calculating we get

$$\begin{aligned}\omega(v, f\dot\gamma) &= -\int_0^1 f(\dot u\cdot\dot\gamma + 2\alpha_0\dot b)dt\\ &= -\int_0^1 f(-u\cdot\ddot\gamma + 2\alpha_0\dot b)dt, &&\text{by integration by parts}\\ &= 0, &&\text{by the definition of } b.\end{aligned}$$

Thus $(**)$ is proven. Collecting loose ends, we find

$$w = a\dot\gamma + u = (a-b)\dot\gamma + v$$

with $a, b \in C^\infty(S^1,\mathbb{R})$ and $v \in V_\gamma^\angle$. Thus every $w \in (\dot\gamma)^\angle$ is contained in $V_\gamma + V_\gamma^\angle$, which proves the first part of (iii).

Since γ is *not* a geodesic we can find a $\varepsilon > 0$ and a family $\{\tilde\gamma_s(t) \mid |s| < \varepsilon\}$ of smooth immersion such that $\tilde\gamma_0(t) = \gamma(t)$ and

$$dL(\tilde\gamma_s)/ds|_{s=0} \neq 0\,,$$

where L denotes the length of a curve. Going through the standard procedure of reparametrization by arclength, but keeping track of the dependence on the parameter on the parameter s, we find a smooth map $G : (-\varepsilon, +\varepsilon)\times S^1 \longrightarrow S^1$ such that $G_s = G(s,\cdot)$ is a diffeomorphism of S^1, $\gamma_s(t) := \tilde\gamma_s(G(s,t))$ is parametrized by arclength and $\gamma_0(t) = \gamma(t)$. It follows that $L(\gamma_s) = L(\tilde\gamma_s)$ and especially

$$dL(\gamma_s)/ds|_{s=0} \neq 0\,.$$

We take the image of the family γ_s in X_∞ as the slice S_γ. Since $(D\Phi_\gamma)(T_\gamma S_\gamma)$ is transversal to the orbit $\mathcal{G}\cdot\Phi(\gamma) = \mathcal{O}_\gamma$ in $\mathbf{g}^*_{\mathrm{reg}}$, $T_\gamma S_\gamma$ is transversal to $V_\gamma + K_\gamma = V_\gamma + V_\gamma^\angle$, thus giving the desired direction and completing the proof of (iii).

Ad (iv) Let w in $K_\gamma^\angle$ be decomposed as $w = a\dot\gamma + u$, where $a \in C^\infty(S^1,\mathbb{R})$ and $u(t)\cdot\dot\gamma(t) = 0$ for all t. As above we define a smooth function

$$b(t) = \frac{1}{2\alpha_0} \int_0^t u(s) \cdot \ddot{\gamma}(s) ds .$$

The assumption implies

$$b(1) - b(0) = \frac{1}{2\alpha_0} \int_0^1 \ddot{\gamma}(s) \cdot u(s) ds = \frac{1}{2\alpha_0} \omega(\dot{\gamma}, u) = 0 ,$$

since $w \in K_\gamma^\angle \subset (\dot{\gamma})^\angle$.

Defining $v := u + b\dot{\gamma}$, we find for $k \in K_\gamma$

$$\begin{aligned} \omega(k, v) &= \omega(k, u) + \omega(k, b\dot{\gamma}) \\ &= \omega(k, u) , && \text{since } V_\gamma \subset K_\gamma^\angle \\ &= \omega(k, w) , && \text{since } V_\gamma \subset K_\gamma^\angle \\ &= 0 , && \text{by assumption.} \end{aligned}$$

Furthermore $\omega(f\dot{\gamma}, v) = 0$ for all f, i.e. $v \in K_\gamma^\angle$ and $v \in V_\gamma^\angle$, short $v \in K_\gamma \cap K_\gamma^\angle$. In the case $\ddot{\gamma} = 0$, we found in (iii) that $V_\gamma + V_\gamma^\angle = T_\gamma X_\infty$ and thus

$$K_\gamma \cap K_\gamma^\angle = V_\gamma^\angle \cap (V_\gamma^\angle)^\angle = (T_\gamma X_\infty)^\angle = N_\gamma .$$

We stop here and set $c = 0$, $z = v$.

In the case $\ddot{\gamma} \neq 0$, we had

$$\omega(\dot{\gamma}, J\dot{\gamma}) = d \neq 0$$

and

$$(V_\gamma + V_\gamma^\angle) \oplus ((J\dot{\gamma}))_{\mathbb{R}} = T_\gamma X_\infty .$$

We define here $c := \omega(v, J\dot{\gamma}) \cdot 1/d \in \mathbb{R}$ and $z := v - c\dot{\gamma}$. A quick calculation yields

$$z \in V_\gamma^\angle \cap (V_\gamma^\angle)^\angle \cap (J\dot{\gamma})^\angle = (T_\gamma X_\infty)^\angle = N_\gamma .$$

Thus in both cases $z \in N_\gamma$ and

$$w = (a - b + c)\dot{\gamma} + z \in V_\gamma + N_\gamma .$$

The second formula $K_\gamma \cap K_\gamma^\angle = R_\gamma + N_\gamma$ follows from the first and (i). □

Remarks 42.

(1) Thinking in terms of the energy

$$H(\gamma) = \frac{1}{2} \int_0^1 \|\dot{\gamma}(t)\|^2 dt ,$$

which is proportional to length on S_γ, the transversal direction to $V_\gamma + V_\gamma^\angle$ (in a non-geodesic point γ) is given by $(\nabla H)_\gamma$, where the gradient must be taken with respect to an appropriate Riemannian metric on X.

(2) The above results (Lemma 40 and Proposition 41) lead to the following geometric picture: Given a non-geodesic, immersive curve γ (parametrized by arclength), the intersection of the orbit $\mathcal{G} \cdot \gamma$ and the fibre Φ_γ is exactly the orbit of γ under the circle group (Rot S^1). The tangents to this space (Rot S^1) $\cdot\, \gamma$ are the degeneration direction of ω restricted to the orbit, while on the other hand this direction R_γ plus the degeneration space N_γ of ω form the degeneration directions of ω restricted to the fibre.

(3) Going through Proposition 41, it follows immediately that (i) and (ii) are valid for X_k, $k \geq 2$. Since V_γ does not sit in $T_\gamma X_k$, but rather in $\overline{T_\gamma X_k}$, the $H^{k-1,2}$-completion, for general γ the assertion (iii) and (iv) can hold only in modified versions for X_k.

We recall that the "reduced space with respect to the orbit $\mathcal{O}_\alpha = \mathcal{G} \cdot \alpha \subset \mathbf{g}^*$" is set-theoretically defined as

$$X_\alpha := \Phi^{-1}(\mathcal{O}_\alpha)/\mathcal{G} \simeq \Phi^{-1}(\alpha)/\mathcal{G}_\alpha ,$$

where $\mathcal{G}_\alpha$ denotes the isotropy group of α. This space carries in "good cases" a symplectic structure $\bar{\omega}$ or a Poisson structure (see [L-S] for the state of the art in the case of proper actions and finite-dimensional groups and manifolds).

In our case we can—for $\alpha \in \mathbf{g}^*_{\mathrm{reg}}$ and strictly positive—assume that α is a constant $\alpha_0 \in \mathbb{R}^{>0}$, since we know a bit about the coadjoint action (Propositions 22 and 24).

The following proposition follows:

Proposition 43. *Let α_0 be in $\mathbb{R}^{>0} \subset \mathbf{g}^*_{\mathrm{reg}}$. Then the reduced space X_{∞,α_0} is*

$$\{\gamma \in X_\infty \mid \gamma \textit{ is immersive and } L(\gamma) = \sqrt{2\alpha_0}\}/\mathcal{G}$$
$$\simeq \{\gamma \in X_\infty \mid \gamma \textit{ is immersive, p.a.l. and } L(\gamma) = \sqrt{2\alpha_0}\}/\sim_{\mathrm{IP}} ,$$

where $\gamma_1 \sim_{\mathrm{IP}} \gamma_2$ iff γ_1 and γ_2 differ only by the choice of the initial point. These sets are in turn isomorphic to the set of unparametrized immersed arcs of length $\sqrt{2\alpha_0}$. Furthermore the "tangent space" of $[\gamma]$ is given by K_γ/R_γ and contains as its degeneration subspace

$$\mathrm{Kern}\, \bar{\omega}_{[\gamma]} = N_\gamma/R_\gamma \cap N_\gamma .$$

Proof. The assertions follow from the fact $\mathcal{G}_{\alpha_0} = \mathrm{Rot}\, S^1$ and the above proposition. □

Proposition 44. *Let α_0 be in $\mathbb{R}^{>0}$ and assume that (M, g) does not contain a closed geodesic of length $\sqrt{2\alpha_0}$. Then the reduced space X_{k,α_0} $(k \geq 2)$ is a "generalized Hilbert manifold", in fact a Hilbert manifold modulo a continuous, almost free circle action equipped with a "symplectic structure".*

Proof. By Proposition 38, $\Phi^{-1}(\alpha_0)$ is a smooth Hilbert manifold. Since all curves in it are immersive $\mathcal{G}_{\alpha_0} = \mathrm{Rot}\, S^1$ acts almost free on it. The "symplectic structure" on X_{k,α_0} is given by ω restricted to $T\Phi^{-1}(\alpha_0)$ since $\dot{\gamma}$, the generator of Rot S^1, is ω-perpendicular to $T_\gamma\Phi^{-1}(\alpha_0)$. □

Remarks 45. Summarizing the facts collected on Φ and $D\Phi$ it is fair to say that in the immersive points the classical momentum geometry sustains despite the facts that
(1) ω is only a degenerate, weakly-symplectic form and
(2) the $\mathcal{G}$-action is neither proper nor differentiable in the strong sense used in the Banach category.

This allows for the hope to study successfully finer properties of the image of Φ as well as of the reduced spaces, having in mind the goal of understanding classical mechanics on the space of unparametrized curves ("closed bosonic strings") in Riemannian manifolds.

4. Further remarks

4.1. Reduced spaces of loop spaces and spaces of closed geodesics

Given a C_l-manifold, i.e. a Riemannian manifold with the property that all geodesics are periodic with the same least period l, it is known that the tangent bundle UM allows a free S^1-action, generated by the energy function on TM, and the quotient $CM := UM/S^1$ is the manifold of oriented geodesics on M. This space is the symplectic reduction of TM equipped with the canonical symplectic structure of the tangent bundle of a Riemannian manifold with respect to the above Hamiltonian circle-action (see [Bess] for all this).

We identify a point $\mathrm{v} \in UM$ with the geodesic γ_v starting in $\pi(\mathrm{v}) \in M$ with initial velocity v. The tangent space to $[\gamma] \in CM$ is the space of normal Jacobi fields along γ, i.e. Jacobi fields J with the property that J and $\nabla J/dt$ are perpendicular to $\dot{\gamma}$ in all points of γ.

Lemma 46. *Given a C_l-manifold (M,g), the space CM is symplectically embedded in the reduced space $X_{l^2/2} = \Phi^{-1}\left(l^2/2\right) / \mathrm{Rot}\, S^1$.*

Proof. The map from CM to $X_{l^2/2}$ is induced by mapping $\mathrm{v} \in UM$ to the geodesic γ_v (with length $l^2/2$). This map is equivariant with respect to the two circle actions involved.

A normal Jacobi field J satisfies

$$(D\Phi)_\gamma(J) = g_{\gamma(t)}\left(\dot{\gamma}(t), \frac{\nabla J}{dt}(t)\right) = 0$$

and J is *not* a multiple of $\dot{\gamma}$ since $J(t)$ is perpendicular to $\dot{\gamma}(t)$ for all t as well. Given now two normal Jacobi fields J_1, J_2 along γ the symplectic form of CM reads:

$$\omega_\gamma^{CM}(J_1,J_2) = g\left(J_1, \frac{\nabla J_2}{dt}\right) - g\left(\frac{\nabla J_1}{dt}, J_2\right),$$

where the RHS is constant in t (see [Bess, p. 58]).

This obviously equals

$$-2\int_0^1 g_{\gamma(t)}\left(\frac{\nabla J_1}{dt}(t), J_2(t)\right) dt = -2\tilde{\omega}_{[\gamma]}([J_1],[J_2]),$$

the symplectic form of $X_{l^2/2}$ (up to the constant normalizing factor -2, see Proposition 44). □

We would like to point out that this observation resulted from a conversation with A. Reznikov.

4.2. The action of the isometry group of (M,g) on the loop space

As we remarked in Lemma 5 in Section 1.3, the isometry group $H := \mathrm{Isom}(M,g)$ acts smoothly on the loop space, the action commutes with the $\mathcal{G}$-action and μ is H-invariant. This provides us with a H-moment map $\Phi_H : X \longrightarrow \mathbf{h}^*$ by contraction:

$$\Phi_H(\gamma)(\eta) := -\mu(\eta_X(\gamma)).$$

Rather than developing the study of this map in general, we will only point out an interesting example:

Lemma 47. *Let (M,g) be $\mathbb{R}^d$ with the flat metric and η be the generator of the rotation in the plane E. Then*

$$\Phi_H(\gamma)(\eta),$$

for γ in the loop space of $\mathbb{R}^d$, equals the signed area of the domain in the plane E circumscribed by the projection of γ to this plane.

Proof. Let us assume for simplicity that η is the rotation in the (1,2)-coordinate plane and denote the projection of γ to this plane by $\gamma_{1,2}$.

A direct computation yields

$$\eta_X(\gamma) = (\gamma_2, -\gamma_1, 0, \ldots, 0),$$

where we identified $T_\gamma X$ with $X \simeq (C^\infty(S^1,\mathbb{R}))^d$. Now

$$\begin{aligned}\Phi_H(\gamma)(\eta) &= -\mu(\eta_X(\gamma)) \\ &= -\frac{1}{2}\int_0^1 g_{\gamma(t)}\,(\eta_X(\gamma)(t),\ \dot\gamma(t))\,dt \\ &= \frac{1}{2}\int_0^1 \gamma_{1,2}^*(x_1 dx_2 - x_2 dx_1)\,,\end{aligned}$$

where x_j are the coordinate functions on $\mathbb{R}^d$. By Stoke's theorem the last expression equals the signed area of the domain circumscribed by $\gamma_{1,2}$ in $\mathbb{R}^2$. □

This seems to be interesting for Riemannian geometry for general d, but especially ties in with symplectic geometry for even d: Restricting to the subgroup $U(d)$ of symplectic isometries implies that we measure only projections to symplectic planes (in fact complex lines). Considering now subsets Ω of $\mathbb{R}^{2d}$ and appropriate sets $\mathcal{L}\Omega$ of closed curves, which especially do not contain multiple coverings of a closed arc in Ω, one can extract invariants like

$$\inf_{\eta\in\mathfrak{u}(d),\|\eta\|=1}\left(\sup_{\gamma\in\mathcal{L}\Omega}\Phi_H(\gamma)(\eta)\right),$$

that should relate to other invariants like symplectic capacities .

4.3. Geometric quantization of loop spaces

Having described the classical aspects of the symplectic structure of loop spaces and the reparametrization action it is quite natural to ask for the ($\mathcal{G}$-equivariant) geometric quantization of it, aiming for a geometrical derivation of "quantized string theory". (In fact this was the main motivation for this work.)

For special targets like flat (Minkowski) space and compact semisimple Lie groups this was attempted the first time by Bowick and Rajeev [Bo-Ra]. Later work of Mickelsson [Mick], Popov and Sergeev [Po-Se] and others clarified (and almost algebraized) the subject. But all these works have a serious gap in the question of how to define measures and square-integrable functions on loop spaces. To leave the realm of partly formal calculations it is absolutely necessary to solve this problem.

In the case of arbitrary target (M,g) a second problem arises, since it is not at all clear how to find polarizations (in the sense of geometric quantization, see e.g. [Wo]) and enough polarized, integrable functions on the loop space.

4.4. Symplectic description of nonlinear Sigma-models

Classical nonlinear Sigma-models (for short σ-models in the sequel) in their simplest form have the set of maps from a flat cylinder to an arbitrary Riemannian manifold as configuration space and the harmonic map equation as its (field) equation.

Motivated by the fact that the geodesics of a Riemannian manifold can be cast in a useful Hamiltonian frame (see below) we look for an analogous interpretation of σ-models. The solution in turn gives a classical "explanation" for the appearance of equivariant Laplacians rather that Laplacians in the (partially heuristic) quantizations of σ-models.

Proposition 48 (Folklore). *Let (M,g) be a Riemannian manifold and $\psi : TM \longrightarrow T^*M$ the isomorphism given by the metric. Denoting the pullback of the canonical 2-form on T^*M by ω^g we have a bijection between:*

(i) integral curves $y(s)$ of the Hamiltonian dynamical system (TM, ω^g, H), where $H(y) = \frac{1}{2}g(y,y)$, with initial conditions $y(0) = (q,p) \in TM \quad (p \in T_qM)$, and

(ii) geodesic curves $\gamma = \gamma(s)$ in (M,g) with initial conditions $\gamma(0) = q \in M$, $\gamma'(0) = p \in T_qM$.

Proof. See e.g. [Kl]. □

Let us denote in this section by X the space of smooth loops in M. Recall that we defined the "good" cotangent bundle $\tilde{T}^*X = C^\infty(S^1, T^*M)$ with fibres $\tilde{T}^*_\gamma X = \Gamma_{C^\infty}(S^1, \gamma^*T^*M)$. We let $\pi : T^*M \longrightarrow M$ be the canonical projection and $\Pi : \tilde{T}^*X \longrightarrow X, \Pi(\alpha) := \pi \circ \alpha$ the induced projection. As in the usual case we find a canonical 1-form on $\tilde{T}^*X$:

$$(\tilde{\theta})_{\alpha_\gamma}(\xi_{\alpha_\gamma}) := \int_0^1 \langle \alpha_\gamma(t), (\Pi_*)_{\alpha_\gamma}(\xi_{\alpha_\gamma})(t) \rangle \, dt,$$

where

$$\alpha_\gamma \in \tilde{T}^*_\gamma X, \quad \xi_{\alpha_\gamma} \in T_{\alpha_\gamma}(\tilde{T}^*X) = \Gamma_{C^\infty}(S^1, \alpha^*_\gamma T(T^*M))$$

and $\langle \ , \ \rangle$ denotes the natural pairing between T_qM and T^*_qM $(q \in M)$. Defining $\tilde{\omega} = -d\tilde{\theta}$ we get a closed, non-degenerate 2-form on $\tilde{T}^*X$.

The metric g gives us a canonical smooth bundle isomorphism $\Psi : TX \longrightarrow \tilde{T}^*X$ over Id_X by mapping $u \in T_\gamma X$ to $\Psi(u)(t) = (\psi \circ u)(t) = g_{\gamma(t)}(u(t)), \cdot)$.

Finally, we define, again quite in analogy to the usual case, $\omega^g := \Psi^*\tilde{\omega}$, the canonical symplectic structure on TX. We would like to remark that Atiyah observed that X is symplectically embedded in $\tilde{T}^*X$ by the 1-form μ—or in our language in TX by the vector field $\zeta(\gamma) = -\dot{\gamma}$.

Mimicking the above proposition we proceed to the natural Hamiltonian $H_0 : TX \longrightarrow \mathbb{R}$, $H_0(u) = \frac{1}{2}\langle u, u\rangle_0$, the fibre norm on TX given by the "L^2-metric".

Lemma 49. *Given a Riemannian manifold (M,g) and its smooth loop space X we have a bijection between*

(i) integral curves $y = y(s)$ of the dynamical system (TX, ω^g, H_0) with initial values

$$y(0) = (\gamma(0), u(0)) = (\gamma_0 g, u_0) \in TX,$$

(ii) *geodesics* $\gamma(s)$ *of the* L^2*-metric on* X *with initial values* $\gamma(0) = \gamma_0$, $\gamma'(0) = u_0$, *and*

(iii) *maps* $f(s,t) : (-\varepsilon, \epsilon) \times S^1 \longrightarrow M$ *with the property that* f *is geodesic in* s *for fixed* t *and* $f(0,t) = \gamma_0(t), \partial f/\partial s(0,t) = u_0(t)$.

Proof.

Step 1. The description of ω^g: Since the tangent bundle of TM splits into the vertical bundle, $V := \mathrm{Kern}(\pi)_*$, and the horizontal bundle H given by Levi-Civita connection ∇ of the metric g we get induced splittings

$$\begin{aligned} T_{u_\gamma}(TX) &= \Gamma_{C^\infty}(S^1, u_\gamma^* TM) \\ &= \Gamma_{C^\infty}(S^1, u_\gamma^* H) \oplus \Gamma_{C^\infty}(S^1, u_\gamma^* V) \\ &=: H_{u_\gamma} \oplus V_{u_\gamma} \end{aligned}$$

(and similarly for $T_{\alpha_\gamma}(\tilde{T}^* X)$).

Given now $w \oplus v$, $w' \oplus v' \in T_{u_\gamma}(TX)$ with w, w' horizontal and v, v' vertical, a computation with coordinates of M yields

$$\omega^g(w \oplus v, w' \oplus v') = \langle w, v' \rangle_0 - \langle w', v \rangle_0 \,.$$

Step 2. Calculation of the Hamiltonian equations: We describe the Hamiltonian field ξ_{H_0} in $u_\gamma \in TX$ as $\xi_{H_0}(u_\gamma) = a \oplus b \in H_{u_\gamma} \oplus V_{u_\gamma}$ and recall

$$\begin{aligned} (dH_0)_{u_\gamma}(w \oplus v) &= \omega^g(a \oplus b, w \oplus v) \\ &= \langle a, v \rangle_0 - \langle w, b \rangle_0 \,. \end{aligned}$$

Standard calculations, using the canonical Riemannian metric on TM and its exponential map, give the result

$$(dH_0)_{u_\gamma}(w \oplus v) = \langle u_\gamma, v \rangle_0$$

and thus

$$\xi_{H_0}(u_\gamma) = u_\gamma \oplus 0 \,.$$

Thus we find the following equation for an integral curve $y(s)$ for ξ_{H_0}:

$$\begin{aligned} y(s) \oplus 0 = \xi_{H_0}(y(s)) &= y'(s) \\ &= (y'(s))_H \oplus (y'(s))_V \,, \end{aligned}$$

i.e. $y(s) = (y'(s))_H$ and $0 = (y'(s))_V$.

Step 3. Bijection between (i) and (iii): Given a solution of (i) we project it to X to find a curve $\gamma(s) = \pi \circ y(s)$ with

$$\gamma'(s) = (y'(s))_H = y(s) \quad \text{and} \quad \gamma''(s) = y'(s) = y(s) \oplus 0 \,,$$

thus $\gamma(s)$ fulfills

$$0 = (\gamma''(s))_V = \nabla_{\partial/\partial s} \gamma'(s) \,,$$

i.e., $\gamma(s)$ is a geodesic in M with respect to the parameter s. Defining $f(s,t) := \gamma(s)(t)$ we find the desired map.

Let now a map $f : (-\varepsilon, \varepsilon) \times S^1 \longrightarrow M$ be given such that the curves $\gamma(s)(t) := f(s,t)$ are geodesics in s for fixed t. Setting $y(s) := \gamma'(s)$ we get a curve in TX. Differentiating it follows that

$$y'(s) = \gamma''(s) = (\gamma''(s))_H \oplus (\gamma''(s))_V ;$$

the second term is zero by assumption on $\gamma(s)$ and thus $(y'(s))_V = 0$. On the other hand, we have $(y'(s))_H = y(s)$ since y is a tangent curve to a curve γ in the base X. Collecting the results we have

$$y'(s) = \xi_{H_0}(y(s)) .$$

The statements on the initial values are obvious.

Step 4. Bijection between (ii) and (iii): To this end we calculate the geodesics of the L^2-metric on X: Let $\gamma : I = [a,b] \longrightarrow X$ be a curve in X and

$$\begin{aligned} E_0(\gamma) &:= \frac{1}{2} \int_a^b \langle \gamma'(s), \gamma'(s) \rangle_0 \, ds \\ &= \frac{1}{2} \int_a^b \left(\int_0^1 g_{\gamma(s)(t)} \left(\gamma'(s)(t), \gamma'(s)(t) \right) dt \right) ds \end{aligned}$$

the energy with respect to the L^2-metric $\langle \ , \ \rangle_0$. Let now U be a vector field along γ with $U(\gamma(a)) = 0 = U(\gamma(b))$ and $\mathrm{Exp}(\varepsilon U)$ the variation of γ in direction U, i.e.,

$$\mathrm{Exp}(\varepsilon U)(t) = \exp^M_{\gamma(s)(t)}(\varepsilon U(s)(t)) .$$

Standard calculations yield the following result: $\gamma = \gamma(s)$ fulfills

$$\begin{aligned} 0 \quad &= \quad \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E_0(\mathrm{Exp}(\varepsilon U)) \quad \forall U \\ &\Longleftrightarrow \nabla_{\partial/\partial s} \gamma'(s) = 0 \quad \forall t . \end{aligned}$$

Thus the bijection between (ii) and (iii) is shown. □

We would like to remark that—as J. Jost pointed out to us—the equivalence between (ii) and (iii) is a special case of a result of D.S. Freed and D. Groisser [F-G, Appendix].

Lemma 49 is not at all sufficient from the point of view of σ-models since it does not involve any derivatives of f with respect of t. To this end we will add a second term to the Hamiltonian, namely

$$V_0(u_\gamma) := -\tfrac{1}{2} \langle \dot\gamma, \dot\gamma \rangle_0 ,$$

the energy of the projection of $u_\gamma \in TX$ to X.

Again a calculation with the exponential map of TM yields the Hamilton field:

$$\xi_{V_0}(y(s)) = 0 \oplus \left(-\nabla_{\partial/\partial t} \left(\frac{\partial}{\partial t}(\pi \circ y(s)) \right) \right)$$

and thus the Hamilton equations:

$$(y'(s))_H = 0,$$
$$(y'(s))_V = -\nabla_{\partial/\partial t} \left(\frac{\partial}{\partial t}(\pi \circ y(s)) \right).$$

Adding H_0 and V_0 up to the new Hamiltonian $H_1 := H_0 + V_0$ we find

Proposition 50. *Given a Riemannian manifold (M, g) and its smooth loop space X we have a bijection between:*

(i) integral curves $y = y(s)$ of the dynamical system (TX, ω^g, H_1) with initial values

$$y(0) = (\gamma(0), u(0)) = (\gamma_0, u_0),$$

and

(ii) harmonic maps from the flat cylinder $(-\varepsilon, \varepsilon) \times S^1$ to the Riemannian manifold (M, g) with initial conditions

$$f(0,t) = \gamma_0(t), \quad \frac{\partial f}{\partial s}(0,t) = u_0(t).$$

Proof. The Hamiltonian equation for H_1 reads:

$$\begin{aligned} y'(s) &= y'(s)_H \oplus y'(s)_V \\ &= \xi_{H_1}(y(s)) \\ &= y(s) \oplus \left(-\nabla_{\partial/\partial t} \left(\frac{\partial}{\partial t}(\pi \circ y(s)) \right) \right), \end{aligned}$$

i.e.,

$$(y'(s))_H = y(s)$$

and

$$(y'(s))_V = -\nabla_{\partial/\partial t} \left(\frac{\partial}{\partial t}(\pi \circ y(s)) \right).$$

Thus, given a solution $y(s)$, the projection $\gamma(s) = \pi \circ y(s)$, fulfills:

$$\gamma'(s) = (y'(s))_H$$

and thus

$$\gamma''(s) = (y'(s))'_H = y'(s)$$

yielding

$$\nabla_{\partial/\partial s}\gamma'(s) = \big(\gamma''(s)\big)_V = \big(y'(s)\big)_V = -\nabla_{\partial/\partial t}\dot{\gamma},$$

i.e.

$$\nabla_{\partial/\partial t}\dot{\gamma} + \nabla_{\partial/\partial s}\gamma' = 0. \tag{$*$}$$

Interpreting γ as a map

$$f : (-\varepsilon, \varepsilon) \times S^1 \longrightarrow M$$

and giving the cylinder the canonical flat metric $(*)$ is equivalent to the harmonic map equation

$$\text{trace}\, \nabla df = 0.$$

Conversely, given a harmonic f, retracing the above argument in the other direction, we find that $y := \gamma'$ is a solution of the dynamical system associated with H_1. □

Remarks 51.

(1) The Hamiltonian

$$H_1(u_\gamma) = \tfrac{1}{2}\langle u_\gamma, u_\gamma\rangle_0 - \tfrac{1}{2}\langle\dot{\gamma},\dot{\gamma}\rangle_0$$

carries a seemingly surprising relative minus-sign in view of the energy

$$E(f) = \frac{1}{2}\int\!\!\int\left(\left|\frac{\partial f}{\partial s}\right|^2 + \left|\frac{\partial f}{\partial t}\right|^2\right) dt\, ds,$$

whose minimizers are the harmonic maps. Its appearance can be explained in terms of classical field theory: Consider the maps from the circle to (M, g) as the set of fields and let $H_1 = H_0 + V_0$ be the Hamiltonian governing the field equation. Now Hamilton's principle asks for extremizing the integral over the Lagrangian $\int_a^b L(\gamma, \gamma', s)ds$ for given $\gamma(s)$ and $\gamma(b)$, but L equals "kinetic energy" − "potential energy" in contrast to the Hamiltonian H, the "total energy", which equals "kinetic energy"+ "potential energy". Thus one should minimize $H_0 - V_0$, i.e. one looks indeed for harmonic maps.

(2) Attempting to quantize the σ-model, many authors consider (Rot S^1)-equivariant Laplacians on loop spaces as the "right" operators (see [Benn] and the references therein), especially since these seem to be more likely to be rigorously definable.

Hoping naively to apply geometric quantization to the triple (TX, ω^g, H_1), a good candidate for the polarized space of sections, the quantization of (TX, ω^g), is $L^2(X)$ with respect to an appropriate measure. The Hamiltonian

$$H_1 = H_0 + V_0 = \tfrac{1}{2}\|u_\gamma\|^2 - \tfrac{1}{2}\|\dot{\gamma}\|^2$$

consists of two terms: the fibre norm H_0 of a tangent bundle, which should give rise to the "normal Laplacian", and the "potential term" V_0, which is the generator of the (Rot S^1)-action on X. This indicates a classical derivation of the appearance of equivariant Laplacians in the quantization of σ-models.

It is a pleasure to thank S. Albeverio, H. Azad, S. Paycha, A. Reznikov, and M. Slupinski for stimulating discussions and especially D. Bennequin and J. Jost for their helpful interest. Finally I would like to acknowledge the support from the SFB 237 "Unordnung und große Fluktuationen" in Bochum during the final stages of this work.

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